

A GENERALIZED SPRINGER DECOMPOSITION FOR D -MODULES ON A REDUCTIVE LIE ALGEBRA

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ABSTRACT. In this paper we show that the (derived or abelian) category of adjoint equivariant D -modules on a complex reductive Lie algebra $\mathfrak{g} = \text{Lie}(G)$, decomposes into orthogonal blocks indexed by conjugacy classes of cuspidal data associated to Levi subalgebras $\mathfrak{l} = \text{Lie}(L)$ of \mathfrak{g} . The blocks of the abelian category are shown to be equivalent to the category of $W(G, L)$ -equivariant D -modules on the center $\mathfrak{z}(\mathfrak{l})$, where $W(G, L) = N_G(L)/L$ is the relative Weyl group. Our result can be thought of as Lusztig's Generalized Springer Correspondence in families. The proof involves analyzing the monad associated to the adjoint functors of parabolic induction and restriction. We indicate how our results relate to the quantum Hamiltonian reduction of Gan-Ginzburg, and the Morse Decompositions of McGerty-Nevins. It is hoped that the same techniques will be fruitful in other related settings, e.g. the Group, Elliptic, Mirabolic, Quantum, and Modular versions of Springer Theory and Character Sheaves. The results of this paper formed part of the authors PhD thesis.

Main results. In his seminal paper [Lus84], Lusztig proved the Generalized Springer Correspondence, which gives a complete description of the category of G -equivariant perverse sheaves on the nilpotent cone $\mathcal{N}_G \subseteq \mathfrak{g} = \text{Lie}(G)$, for a reductive group G . This description involves decomposing into blocks indexed by *cuspidal data*: triples $(L, \mathcal{O}, \mathcal{E})$ of a Levi subgroup L of G , a nilpotent orbit \mathcal{O} for L , and a certain L -equivariant local system \mathcal{E} on \mathcal{O} . Each block is then identified with the category of representations of the relative Weyl group $W(G, L) = N_G(L)/L$.

One of the main results of this paper is that Lusztig's description of equivariant perverse sheaves on the nilpotent cone extends to a description of all G -equivariant D -modules on \mathfrak{g} :

Theorem 0.1. *There is an orthogonal decomposition of the abelian category $\mathbf{M}^G(\mathfrak{g})$ of G -equivariant D -modules on \mathfrak{g} into blocks $\mathbf{M}(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$, indexed by G -conjugacy classes of cuspidal data, such that*

$$\mathbf{M}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})} \simeq \mathbf{M}^{W(G, L)}(\mathfrak{z}(\mathfrak{l})) \simeq \mathfrak{D}(\mathfrak{z}(\mathfrak{l})) \rtimes W(G, L) - \text{mod}.$$

Here $\mathfrak{z}(\mathfrak{l})$ denotes the center of the Lie algebra \mathfrak{l} which it carries a natural action of the finite group $W(G, L)$.¹ If we restrict to the subcategory of D -modules with support on the nilpotent cone (which can be identified with a category of perverse sheaves via the Riemann-Hilbert correspondence), the corresponding blocks restrict to the category of representations of $W(G, L)$, which is Lusztig's Generalized Springer Correspondence.

Example 0.2. In the case $G = GL_n$, it is known that there is a unique cuspidal datum up to conjugacy, corresponding to a maximal torus of T_n of GL_n . Thus we have:

$$\mathbf{M}^{GL_n}(\mathfrak{gl}_n) \simeq \mathbf{M}^{S_n}(\mathfrak{t}_n),$$

where $\mathfrak{t}_n = \text{Lie}(T_n) \simeq \mathbb{C}^n$.

¹In fact, in the cases when L carries a cuspidal local system, $W(G, L)$ is a Coxeter group and $\mathfrak{z}(\mathfrak{l})$ its reflection representation.

The block decomposition of $\mathbf{M}^G(\mathfrak{g})$ extends to a decomposition of the derived category $\mathbf{D}^G(\mathfrak{g})$ of G -equivariant complexes of D -modules on \mathfrak{g} .

Theorem 0.3. *There is an orthogonal decomposition*

$$\mathbf{D}^G(\mathfrak{g}) = \bigoplus^{\perp} \mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})},$$

indexed by G -conjugacy classes of cuspidal data $(L, \mathcal{O}, \mathcal{E})$.

The derived orthogonal decomposition of Theorem 0.3 is very strong. Concretely, it means that

- (1) every object $\mathfrak{M} \in \mathbf{D}^G(\mathfrak{g})$ can be written as a direct sum $\mathfrak{M} \simeq \bigoplus \mathfrak{M}_{(L, \mathcal{O}, \mathcal{E})}$ where $\mathfrak{M}_{(L, \mathcal{O}, \mathcal{E})} \in \mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$, and
- (2) the subcategories $\mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$ are pairwise orthogonal, in the sense that there are no non-zero morphisms in either direction between the subcategories associated to non-conjugate cuspidal data.

The description of the blocks $\mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$ is less explicit. Similar to the abelian case, we can identify $\mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$ with the category of modules for a monad acting on $\mathbf{D}^{Z^\circ(L)}(\mathfrak{z}(\mathfrak{l}))$ (where $Z^\circ(L)$ is the connected center of L), however, this monad is not equivalent to the monad representing the action of $W(G, L)$ as in the abelian case. In concrete terms, for each cuspidal datum $(L, \mathcal{O}, \mathcal{E})$, we must choose in addition a parabolic subgroup P of G containing L as a Levi factor. Given this data, we can construct certain differential graded algebra $\mathbf{H}_{(P, L, \mathcal{O}, \mathcal{E})}$ such that there is an equivalence:

$$\mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})} \simeq \mathbf{H}_{(P, L, \mathcal{O}, \mathcal{E})} - \mathbf{dgMod}.$$

The algebra $\mathbf{H}_{(P, L, \mathcal{O}, \mathcal{E})}$ is defined in terms of the relative Borel-Moore homology of a variant of the Steinberg variety associated to the data $(P, L, \mathcal{O}, \mathcal{E})$. Its cohomology algebra can be described as follows:

$$H^\bullet(\mathbf{H}_{(P, L, \mathcal{O}, \mathcal{E})}) \simeq (\Lambda_{\mathfrak{z}(\mathfrak{l})} \otimes \mathfrak{D}_{\mathfrak{z}(\mathfrak{l})}) \rtimes W(G, L).$$

where $\Lambda_{\mathfrak{z}(\mathfrak{l})}$ denotes the graded algebra $\text{Sym}(\mathfrak{z}(\mathfrak{l})[1])$ (this is an exterior algebra when considered as an ungraded algebra).² In general though, the differential graded algebra $\mathbf{H}_{(P, L, \mathcal{O}, \mathcal{E})}$ is not formal, and it is not clear if it admits a description purely in terms of the group $W(G, L)$ together with its action on $\mathfrak{z}(\mathfrak{l})$.³

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²Note that the equivariant derived category $\mathbf{D}^{Z^\circ(L)}(pt)$ for the torus $Z^\circ(L)$ acting on a point is equivalent to $\Lambda_{\mathfrak{z}(\mathfrak{l})} - \mathbf{dgMod}$.

³There is a natural subcategory of $\mathbf{D}^G(\mathfrak{g})$ for which formality does hold, namely the derived character (or orbital) sheaves - see Theorem 1.1.

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1. BACKGROUND AND MOTIVATION

1.1. Character Sheaves, Orbital Sheaves, and Harmonic Analysis. The study of equivariant differential equations (or perverse sheaves) on reductive Lie groups and Lie algebras has a rich history. Given a complex reductive group G , Harish-Chandra showed that any invariant eigendistribution on (a real form of) G satisfies a certain system of differential equations [HC64]. Hotta and Kashiwara later reinterpreted Harish-Chandra's system as an adjoint equivariant D -module \mathfrak{M}_χ on the Lie algebra \mathfrak{g} of G (depending on a parameter $\chi \in \text{Spec } \mathbb{C}(\mathfrak{g})^G$) [HK84]. It was shown in *loc. cit.* that the Fourier transform of \mathfrak{M}_0 can be identified with the *Springer sheaf* \mathfrak{S}_0 (via the Riemann-Hilbert correspondence) which appears in the interpretation of Springer's correspondence [Spr78] due to Lusztig [Lus81], Hotta [Hot81] and Borho–MacPherson [BM83].

The D -module \mathfrak{M}_0 is the canonical example of a *character sheaf*⁴, and the D -module \mathfrak{S}_0 is the canonical example of an *orbital sheaf*. Lusztig developed the theory of character sheaves in the series of papers [Lus85] with spectacular applications to the representation theory of finite groups of Lie

⁴Here, we freely (ab)use the Riemann-Hilbert correspondence by identifying the D -module \mathfrak{M}_0 with its corresponding perverse sheaf.

type. The theory has been reframed in various settings since then; in particular, the D -module approach of Ginzburg [Gin89] [Gin93], (revisited by Mirković in the Lie algebra setting [Mir04]) has been particularly influential in this paper.

Following Ginzburg, we define the subcategory $\mathbf{DCh}_G(\mathfrak{g})$ of *derived character sheaves* (or *admissible complexes*) to consist of objects of $\mathbf{D}^G(\mathfrak{g})$ for which $\mathrm{Sym}(\mathfrak{g})^G \subseteq \mathfrak{D}_{\mathfrak{g}}$ acts locally finitely. Similarly, we have the category $\mathbf{DOrb}_G(\mathfrak{g})$ of *derived orbital sheaves*, generated by objects supported on single conjugacy classes. The subcategories $\mathbf{DCh}_G(\mathfrak{g})$ and $\mathbf{DOrb}_G(\mathfrak{g})$ are interchanged by Fourier transform. Note that the category $\mathbf{DCh}_G(\mathfrak{g})$ decomposes as an orthogonal sum of subcategories $\mathbf{DCh}_{G,\theta}(\mathfrak{g})$ according to the generalized eigenvalues $\theta \in \mathrm{Spec} \mathrm{Sym}(\mathfrak{g})^G$, and there is a similar decomposition for derived orbital sheaves. Thus, understanding derived character sheaves for a Lie algebra reduces to understanding the category $\mathbf{D}^G(\mathcal{N}_G)$ of complexes with support on the nilpotent cone.

One difference between this paper and the work of previous authors is that our results are about the entire category $\mathbf{M}^G(\mathfrak{g})$ (or $\mathbf{D}^G(\mathfrak{g})$) as opposed to the subcategory of (derived) character sheaves (or orbital sheaves). This difference is conceptually important as it allows one to do harmonic analysis on the category $\mathbf{M}^G(\mathfrak{g})$, in the following sense. Given a class function (or distribution) on a Lie group \mathcal{G} , it is often useful to express it in terms of irreducible characters of \mathcal{G} ; when \mathcal{G} is abelian, this is understood in terms of the theory of Pontryagin duality. On the other hand, if \mathcal{G} is non-abelian, there is a beautiful and intricate theory of harmonic analysis due to Harish-Chandra. Very roughly, the problem of understanding the spectrum of \mathcal{G} (or at least its continuous parts) reduces to understanding characters of abelian subgroups of \mathcal{G} together with the action of finite symmetry group. In a similar way, Theorem 0.1 gives a way of understanding a “class D -module” $\mathfrak{M} \in \mathbf{M}^G(\mathfrak{g})$ in terms of class D -modules on abelian Lie algebras $\mathfrak{z}(\mathfrak{l})$, together with the action of the finite group $W(G, L)$. An application of this idea will be presented in forthcoming work of the author with D. Ben-Zvi and D. Nadler on the cohomology of character varieties.

One important feature of the subcategory of derived character (or orbital) sheaves is that the blocks are formal.

Theorem 1.1 (Rider [Rid13], Rider-Russell (in progress)). *There is a t -exact equivalence of differential graded categories:*

$$\mathbf{D}^G(\mathcal{N}_G)_{(L, \mathcal{O}_L, \mathcal{E}_L)} \simeq \Lambda_{\mathfrak{z}(\mathfrak{l})} \rtimes W(G, L) - \mathbf{dgMod},$$

A new proof of Theorem 1.1 will be presented in a sequel to this paper.

Remark 1.2.

- (1) In [Rid13], the algebra $S_{\mathfrak{z}(\mathfrak{l})} := \mathrm{Sym}(\mathfrak{z}(\mathfrak{l})[-2])$ appears instead of $\Lambda_{\mathfrak{z}(\mathfrak{l})}$; these two algebras are Koszul dual.
- (2) A variant of Theorem 1.1 involves studying the category $\mathbf{D}^{G \times \mathbb{C}^\times}(\mathcal{N}_G)$ of complexes which carry an additional \mathbb{C}^\times equivariance. One expects from [Lus88] that the algebra $\Lambda_{\mathfrak{z}(\mathfrak{l})} \rtimes W(G, L)$ will be replaced by a graded Hecke algebra.

1.2. Harish-Chandra Homomorphism and Hamiltonian Reduction. Our results also provide new points of view on the Harish-Chandra homomorphism of Levasseur-Stafford [LS95] [LS96]. Recall that for any complex reductive group G , there is a canonical G -equivariant D -module $\mathfrak{D}_{\mathfrak{g}/G} := \mathfrak{D}_{\mathfrak{g}}/\mathfrak{D}_{\mathfrak{g}\mathrm{ad}(\mathfrak{g})} \in \mathbf{M}^G(\mathfrak{g})$. The object $\mathfrak{D}_{\mathfrak{g}/G}$ is important as it represents the functor of quantum Hamiltonian reduction:

$$\mathbb{H} := \mathrm{Hom}_{\mathbf{M}^G(\mathfrak{g})}(\mathfrak{D}_{\mathfrak{g}/G}, -) : \mathbf{M}^G(\mathfrak{g}) \rightarrow \mathfrak{D}(\mathfrak{g}/G) - \mathbf{mod},$$

where $\mathfrak{D}(\mathfrak{g}/G) = \text{End}_{\mathbf{M}^G(\mathfrak{g})}(\mathfrak{D}_{\mathfrak{g}/G}) = (\mathfrak{D}_{\mathfrak{g}/G})^G$.

There is also a canonical cuspidal datum $(T, \{0\}, \mathbb{C})$ corresponding to a maximal torus T of G with its unique cuspidal local system. Thus according to Theorem 0.1 there is an equivalence

$$(1) \quad \mathbf{M}^G(\mathfrak{g})_{Spr} := \mathbf{M}^G(\mathfrak{g})_{(T, \{0\}, \mathbb{C})} \simeq \mathbf{M}(\mathfrak{t})^W = \mathfrak{D}(\mathfrak{t}) \rtimes W - \text{mod},$$

where $W = W(G, T)$ is the Weyl group.

The results of Hotta-Kashiwara [HK84], understood in the context of Theorem 0.1, state that $\mathfrak{D}_{\mathfrak{g}/G}$ corresponds to the object $\mathfrak{D}_{\mathfrak{t}} \in \mathbf{M}(\mathfrak{t})^W$, under the equivalence 1. It follows that we have an isomorphism

$$(\mathfrak{D}_{\mathfrak{g}/G})^G \simeq \text{End}_{\mathbf{M}^G(\mathfrak{g})}(\mathfrak{D}_{\mathfrak{g}/G}) \simeq \text{End}_{\mathbf{M}^W(\mathfrak{t})}(\mathfrak{D}_{\mathfrak{t}}) \simeq (\mathfrak{D}_{\mathfrak{t}})^W.$$

Such an isomorphism is constructed in [LS95] using the radial parts map.

Moreover, the construction of the equivalence in Theorem 0.1 gives rise to the following commutative diagram:

$$\begin{array}{ccc} \mathbf{M}^G(\mathfrak{g}) & \xrightarrow{\widetilde{\text{res}}} & \mathfrak{D}_{\mathfrak{t}} \rtimes W - \text{mod} \\ \downarrow \mathbb{H} & & \downarrow (-)^W \\ \mathfrak{D}(\mathfrak{g}/G) - \text{mod} & \longrightarrow & \mathfrak{D}(\mathfrak{t})^W - \text{mod} \end{array}$$

Here, res denotes the functor of parabolic restriction to T (for some choice of Borel subgroup B containing T). This functor takes values in the category $\mathbf{M}^T(\mathfrak{t}) = \mathbf{M}(\mathfrak{t})$; however, any object of the form $\text{res}(\mathfrak{M})$ carries a canonical W -action;⁵ the functor $\widetilde{\text{res}}$ takes an object $\mathfrak{M} \in \mathbf{M}^G(\mathfrak{g})$ to the object $\text{res}(\mathfrak{M})$ together with its W -equivariant structure, considered as a $\mathfrak{D}_{\mathfrak{t}} \rtimes W$ -module. The vertical arrow on the right is the standard Morita equivalence between $\mathfrak{D}_{\mathfrak{t}} \rtimes W$ and its spherical subalgebra $(\mathfrak{D}_{\mathfrak{t}})^W$.

In particular, the kernel of the functor \mathbb{H} is the same as the kernel of the functor res . It would be interesting to understand the analogue of the Hamiltonian reduction functor for the other blocks of $\mathbf{M}^G(\mathfrak{g})$, corresponding to cuspidal data, or equivalently, a version of the Hotta-Kashiwara Theorem for other blocks

1.3. Mirabolic D -modules and Cherednik Algebras. Consider the case $G = GL_n$ and let $V = \mathbb{C}^n$ denote the standard representation. The category $\mathbf{M}_c^G(\mathfrak{g} \times V)$ of *Mirabolic D -modules* has a very similar behavior to $\mathbf{M}^G(\mathfrak{g})$, and has been extensively studied, see e.g [EG02] [GG06] [BG12] [FGT09]. As in Subsection 1.2 there is a Mirabolic D -module

$$\mathfrak{D}_{(\mathfrak{g} \times V)/G}^c := \mathfrak{D}_{\mathfrak{g} \times V}^c / \mathfrak{D}_{\mathfrak{g} \times V}^c \text{ad}(\mathfrak{g}),$$

which represents the quantum Hamiltonian Reduction functor.

In particular, it was shown in [GG06] that the ring of endomorphisms of $\mathfrak{D}_{(\mathfrak{g} \times V)/G}^c$ is isomorphic to the spherical subalgebra of the Rational Cherednik Algebra. By analogy with the results of this paper, one expects there to be induction and restriction functors, which cut up the Mirabolic category into pieces, the largest of which can be identified with modules for the rational Cherednik algebra.

All this suggests a relationship between our results and the work of McGerty and Nevins [MN14]. In *loc. cit.*, the authors describe a recollement of the category of twisted D -modules on a quotient stack X/G . The subcategories involved in the recollement are cut out by a stratification of the

⁵This follows from Theorem 6.3

cotangent bundle $T^*(X/G)$.⁶ In particular, their results apply to the Mirabolic category. The fact that there is a recollement rather than an orthogonal decomposition in general should not be surprising given the proof of Theorem 0.3 which makes essential use of the semisimplicity of the group algebra of $W(G, L)$.

1.4. Trigonometric, Elliptic, and Quantum Versions. It is natural to consider the equivalent problem on the group, namely an analogue of Theorems 0.1 and 0.3 for the categories $\mathbf{D}^G(G)$ and $\mathbf{M}^G(G)$. These results will appear in a sequel to this paper. (the proof needs to be adjusted due to the lack of the Fourier transform in the group setting).

In the group setting, the nilpotent orbit \mathcal{O}_L needs to be replaced by an *isolated conjugacy class* of L . Unipotent conjugacy classes are examples of such, but in general, there are non-unipotent isolated classes. One expects the blocks of the abelian category to take the form:

$$\mathbf{M}(Z^\circ(L))^{W(G, L)} \simeq \mathfrak{D}_{Z^\circ(L)} \rtimes W(G, L) - \text{mod}.$$

We refer to the group setting as “trigonometric”, because the algebra $\mathfrak{D}_{Z^\circ(L)} \rtimes W(G, L)$ appearing in above is a specialization of the trigonometric Cherednik algebra (thus the Lie algebra setting would be “rational”).

There are (at least) two more versions of the same theorem one could ask for. To make sense of the first, we must make use of the language of stacks. It is known that the stack \mathfrak{g}/G (respectively G/G) can be identified with the moduli stacks of semistable degree 0 G -bundles on a genus 1 curve with a singe cusp (respectively, node). Given a smooth, and proper curve E of genus 1, let \mathcal{G}_E denote the moduli stack of semistable G -bundles of degree 0 on E . One might expect that the category $\mathbf{M}(\mathcal{G}_E)$ behaves much like $\mathbf{M}(G/G) = \mathbf{M}^G(G)$ or $\mathbf{M}(\mathfrak{g}/G) = \mathbf{M}^G(\mathfrak{g})$.⁷ The analogue of Theorem 0.1 in this setting would express the blocks of $\mathbf{M}(\mathcal{G}_E)$ in terms of categories of the form $\mathbf{M}(\text{Pic}(E) \otimes_{\mathbb{Z}} X_*(Z^0(L)))$. This is the subject of some current work in progress of the author with D. Fratila and P. Li.

In the second version, we consider the ring of quantum differential operators \mathfrak{D}_G^q , as studied by Jordan [Jor09]. There is a category $\mathbf{M}_G^q(G)$ of “strongly equivariant \mathfrak{D}_G^q -modules”. This category appears as the value of a certain TQFT on a closed 2-manifold of genus 1 (see [BZBJ15]). One might also hope enjoys similar properties. In this case, the blocks of $\mathbf{M}_G^q(G)$ would be of the form $\mathfrak{D}_{Z^\circ(L)}^q \rtimes W(G, L)$, where $\mathfrak{D}_{Z^\circ(L)}^q$ is the ring of q -difference operators on the torus $Z^\circ(L)$. Such results would allow for a new interpretation of the ring \mathfrak{D}_G^q in terms of q -difference operators. This is the subject of work in progress with D. Ben-Zvi and D. Jordan.

1.5. Overview of some key ideas. Let us first explain how to view the blocks $\mathbf{D}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$ as subcategories of $\mathbf{D}^G(\mathfrak{g})$. Lusztig defined a stratification of \mathfrak{g} , with strata $\mathfrak{g}_{(L, \mathcal{O})}$ indexed by nilpotent orbits of Levi subgroups. We write $\mathfrak{g}_{(L)}$ for the union of strata $\mathfrak{g}_{(L, \mathcal{O})}$ with fixed Levi L , $\mathfrak{g}_{\leq(L)}$ for the union of $\mathfrak{g}_{(M, \mathcal{O})}$ with M conjugate to a subgroup of L , and similarly we have unions of strata $\mathfrak{g}_{\leq(L)}$, $\mathfrak{g}_{\leq(L)}$, etc.

We define the full subcategory $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ to consist of objects \mathfrak{M} such that the singular support of \mathfrak{M} is contained in (the closed conic subset) $\mathfrak{g}_{\leq(L)} \times \mathfrak{g}_{\leq(L)}$ of $\mathfrak{g} \times \mathfrak{g} \simeq T^*\mathfrak{g}$. The subcategory $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ is defined similarly; the subcategory $\mathbf{D}^G(\mathfrak{g})_{\leq(L)}$ can be defined to be the left orthogonal to $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ in $\mathbf{D}^G(\mathfrak{g})$, and $\mathbf{D}^G(\mathfrak{g})_{(L)}$ can be defined as the left orthogonal to $\mathbf{D}^G(\mathfrak{g})_{\leq(L)}$ in $\mathbf{D}^G(\mathfrak{g})_{\leq(L)}$.

⁶In the present paper, the blocks $\mathbf{M}^G(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$ are also associated to a stratification of the cotangent bundle $T^*\mathfrak{g}$.

⁷This observation places the current paper in the context of the Geometric Langlands Program, where one studies D -modules on the moduli space of G -bundles on curves.

Given a parabolic subgroup P of G , and $L \subseteq P$ a Levi factor, we have adjoint functors:

$$\mathbf{Res}_{P,L}^G : \mathbf{D}^G(\mathfrak{g}) \xrightleftharpoons{\quad} \mathbf{D}^L(\mathfrak{l}) : \mathbf{Ind}_{P,L}^G,$$

We will show that the functors $\mathbf{Ind}_{P,L}^G$ and $\mathbf{Res}_{P,L}^G$ preserve bounded coherent complexes of D -modules, and preserve the heart of the t -structure (thus they define exact functors between the corresponding abelian categories).

Parabolic induction and restriction functors satisfy the following Mackey property. Let Q be another parabolic subgroup of G with Levi factor M . Given $\mathfrak{M} \in \mathbf{D}^L(\mathfrak{l})$, the object

$$\mathbf{Res}_{Q,M}^G \mathbf{Ind}_{P,L}^G(\mathfrak{M}) \in \mathbf{D}^M(\mathfrak{m})$$

can be written as an iterated extension of objects

$$\mathbf{Ind}_{M \cap {}^\psi P, M \cap {}^\psi L}^M \mathbf{Res}_{Q \cap {}^\psi L, M \cap {}^\psi L}^{{}^\psi L} {}^\psi *$$

as $\psi \in G$ ranges over representatives of double cosets $Q \backslash G / P.s$

We will show that the subcategory $\mathbf{D}^G(\mathfrak{g})_{\nless(L)}$ is equal to the kernel of the functor $\mathbf{Res}_{P,L}^G$. Similarly, $\mathbf{D}(\mathfrak{g})_{\nless(L)}$ is the subcategory of objects \mathfrak{M} such that $\mathbf{Res}_{P,L}^G(\mathfrak{M})$ is cuspidal (i.e. any further parabolic restriction to a proper sub-Levi of L is zero). Note that the definition of $\mathbf{D}^G(\mathfrak{g})_{\nless(L)}$ does not refer to any parabolic subgroup; it follows that the kernel of parabolic restriction is independent of the choice of parabolic (containing the fixed Levi L).

The Mackey Theorem implies that parabolic induction and restriction restrict to an adjunction

$$\mathbf{Res}_{P,L}^G|_{\nless(L)} : \mathbf{D}^G(\mathfrak{g})_{\nless(L)} \xrightleftharpoons{\quad} \mathbf{D}^L(\mathfrak{l})_{\text{cusp}} : \mathbf{Ind}_{P,L}^G|_{\text{cusp}},$$

Moreover, the corresponding monad $\mathbf{St} := \mathbf{Res}_{P,L}^G \mathbf{Ind}_{P,L}^G$ is an iterated extension of the functors $w_* : \mathbf{D}^L(\mathfrak{l})_{\text{cusp}}$ as w ranges over the relative Weyl group $W(G, L)$. In general these extensions are not split; however, we can show that the action of the monad \mathbf{St} on the abelian category $\mathbf{M}(\mathfrak{l})_{\text{cusp}}$ agrees with the group monad:

$$W(G, L)_* : \mathfrak{M} \mapsto \bigoplus_{w \in W(G, L)} w_* \mathfrak{M}.$$

This is the most important observation of the paper. It follows from this observation that if $\mathfrak{M} \in \mathbf{M}(\mathfrak{g})_{\nless(L)}$, then the object $\mathbf{Ind}_{P,L}^G \mathbf{Res}_{P,L}^G(\mathfrak{M})$ carries an action of $W(G, L)$ by endomorphisms, and thus $\mathbf{Ind}_{P,L}^G \mathbf{Res}_{P,L}^G(\mathfrak{M})$ decomposes as a direct sum according to the irreducible representations of $W(G, L)$. Moreover, the summand corresponding to the trivial representation is the precisely the projection of \mathfrak{M} on to the subcategory $\mathbf{M}^G(\mathfrak{g})_{(L)}$. This procedure can be iterated to decompose any object $\mathfrak{M} \in \mathbf{M}^G(\mathfrak{g})$ into summands $\mathfrak{M}_L \in \mathbf{M}^G(\mathfrak{g})_{(L)}$.

1.6. Acknowledgments.

2. PRELIMINARIES

2.1. Stable ∞ -categories. In this paper we will work in the setting of \mathbb{C} -linear, stable, presentable, ∞ -categories, as developed by Jacob Lurie [Lur06] [Lur11], or alternatively, pretriangulated differential graded categories (see [Coh13] for the relationship between the two theories). We refer the reader to [BZFN10] or [Gai12] for an overview of the main results and techniques, and directions towards further references. Below we outline some key properties.

To each \mathbb{C} -linear, stable, presentable ∞ -category \mathcal{C} , the homotopy category $h\mathcal{C}$ is a triangulated category; one thinks of \mathcal{C} as an enhancement of the triangulated category $h\mathcal{C}$. For much of this paper, the reader may replace stable ∞ -categories with their homotopy categories without any loss of understanding. However, the extra structure of these enhancements allow for much cleaner and more natural proofs of many of the results in this paper.

Definition 2.1. Let \mathcal{C}, \mathcal{D} be stable, presentable, \mathbb{C} -linear ∞ -categories. Given objects $c, d \in \mathcal{C}$, we have a complex $R\mathrm{Hom}(c, d)$ of morphisms from c to d .

- (1) A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called *continuous* if it preserves all small colimits.
- (2) An object $c \in \mathcal{C}$ is called *compact* if the functor $R\mathrm{Hom}(c, -)$ is continuous (it is equivalent to check that the functor preserves direct sums).
- (3) We say F is *proper* if it sends compact objects to compact objects.

The collection of stable presentable ∞ -categories forms an $(\infty, 1)$ -category $\mathbf{St}_{\mathbb{C}}^{\mathrm{cont}}$, in which the morphisms are continuous functors. We also have ∞ -categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D})$, of functors and continuous functors respectively; both of these categories are themselves stable, presentable, and \mathbb{C} -linear.

A category \mathcal{C} is called compactly generated if there is a subset \mathcal{S} of compact objects in \mathcal{C} such that the right orthogonal to \mathcal{S} vanishes. Given $\mathcal{C} \in \mathbf{St}_{\mathbb{C}}^{\mathrm{cont}}$, we write \mathcal{C}_c for the subcategory of compact objects. We can recover \mathcal{C} from \mathcal{C}_c as the Ind-category: $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_c)$. All the categories arising in this paper will be compactly generated.

Example 2.2. Given a differential graded (dg) algebra A , we have the stable ∞ -category of perfect complexes $\mathbf{Perf}(A)$, and $A\text{-}\mathbf{dgMod} = \mathrm{Ind}(\mathbf{Perf}(A))$ is the category of unbounded complexes of A -modules. In the special case $A = \mathbb{C}$, we write $\mathbf{Vect} := \mathbb{C}\text{-}\mathbf{dgMod}$.

The category $\mathbf{St}_{\mathbb{C}}^{\mathrm{cont}}$ carries a monoidal product \otimes , which is characterized by the property that continuous functors from $\mathcal{C} \otimes \mathcal{D}$ to \mathbf{Vect} are the same thing as functors from $\mathcal{C} \times \mathcal{D}$ to \mathbf{Vect} which are continuous in each argument separately. Given dg algebras A and B , we have $A\text{-}\mathbf{dgMod} \otimes B\text{-}\mathbf{dgMod} = A \otimes B\text{-}\mathbf{dgMod}$.

Proposition 2.3. Suppose $\mathcal{C} \in \mathbf{St}_{\mathbb{C}}^{\mathrm{cont}}$ is compactly generated category. Then \mathcal{C} is dualizable, with dual $\mathcal{C}' := \mathrm{Ind}(\mathcal{C}_c^{\mathrm{op}})$.

Note that if \mathcal{C} is compactly generated then we have an equivalence

$$\mathrm{Fun}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D}) = \mathcal{C}' \otimes \mathcal{D}.$$

Suppose $\mathcal{C}, \mathcal{D} \in \mathbf{St}_{\mathbb{C}}^{\mathrm{cont}}$ are compactly generated, and

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

are an adjoint pair of functors (i.e. L is left adjoint to R). Then R is continuous (i.e. R preserves small colimits) if and only if L is proper (i.e. L sends compact objects to compact objects). In that case, there is an adjunction

$$L_c : \mathcal{C}_c \rightleftarrows \mathcal{D}_c : R_c.$$

Conversely, if

$$L_c : \mathcal{C}_c \rightleftarrows \mathcal{D}_c : R_c$$

is an adjoint pair of functors between the subcategories of compact objects, then we have an adjunction:

$$\mathrm{Ind}(L_c) : \mathcal{C} \rightleftarrows \mathcal{D} : \mathrm{Ind}(R_c).$$

Definition 2.4. We say that a diagram

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

in $\mathbf{St}_{\mathcal{C}}^{cont}$ is a *continuous adjunction* if L is left adjoint to R , and R is continuous (equivalently, L is proper).

The following notation will be useful for us later.

Definition 2.5. A *filtration* of an object a in a stable category \mathcal{C} , indexed by a poset (I, \leq) , is a functor

$$(I, \leq) \rightarrow \mathcal{C}/a \\ , i \mapsto (a_{\leq i} \mapsto a).$$

In the cases of interest to us, I will be a finite poset with a maximal element i_{max} , and we demand in addition that $a_{\leq i_{max}} \rightarrow a$ is an isomorphism. Given a closed subset Z of I (i.e. if $j \in Z$ and $i \leq j$, then $i \in Z$), we define a_Z to be $\text{colim}_{j \in Z} a_{\leq j}$. For any subset J of I , we define a_J to be the cone of $a_{< J} \rightarrow a_{\leq J}$. Note that there is no requirement for the maps to be injective (indeed, it is not clear what this would mean in general).

2.2. The Barr-Beck(-Lurie) theorem and colocalizations. In this subsection we will consider two contexts in parallel:

- (1) Grothendieck abelian categories;
- (2) Compactly generated, stable, presentable ∞ -categories.

All the results will apply to either setting, and we will use the same notation for both.

Suppose that

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R,$$

is an adjunction (L is left adjoint to R), where either:

- (1) \mathcal{C} and \mathcal{D} are Grothendieck abelian categories, and the functors L and R are exact and preserve direct sums; or,
- (2) \mathcal{C} and \mathcal{D} are compactly generated, stable, presentable ∞ -categories, and the functors L and R are continuous.

Remark 2.6. If we are in context (2), and in addition, the categories \mathcal{C} and \mathcal{D} carry a t -structure which the functors R and L preserve, then taking the heart of the t -structure gives an example of context (1).

Let $T = RL$ denote the corresponding monad. Note that for any object $d \in \mathcal{D}$, $R(d)$ is a module for T . Thus we have the following diagram:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{R} & \mathcal{C} \\ & \searrow \tilde{R} & \nearrow \\ & \mathcal{C}^T & \end{array}$$

Definition 2.7. A functor $R : \mathcal{D} \rightarrow \mathcal{C}$ is called *conservative* if whenever $R(x) \simeq 0$ then $x \simeq 0$.⁸

⁸The usual definition of a conservative functor is a functor G such that if $G(\phi)$ is an isomorphism, then ϕ is an isomorphism. This definition is equivalent to the one above, in our context, by considering the cone of ϕ .

Theorem 2.8 (Barr-Beck [BW85], [Lur12] Theorem 6.2.0.6). *If R is conservative, $\tilde{R} : \mathcal{D} \rightarrow \mathcal{C}^T$ is an equivalence.*

Remark 2.9. It will be important to understand how to compute the inverse functor $\tilde{R}^{-1} : \mathcal{C}^T \rightarrow \mathcal{D}$. Given a T -module c in \mathcal{C} , we have a simplicial diagram in \mathcal{D} :

$$(2) \quad Lc \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} LRLc \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} LRLRLc \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \dots$$

The object $\tilde{R}^{-1}(y)$ is given by the colimit of Diagram 2. Indeed, applying the functor R to diagram 2 is the canonical simplicial resolution of $y \in \mathcal{C}$. (In context (1), we only need to consider the first two terms of simplicial diagram 2, which is a coequalizer diagram. The object $\tilde{R}^{-1}(y)$ is given by the coequalizer.)

Now we consider the case where R is not necessarily conservative. Let \mathcal{K} denote the kernel of R , i.e. the full subcategory of \mathcal{D} consisting of objects d such that $R(d) \simeq 0$. Let \mathcal{Q} denote the category of modules \mathcal{C}^T .

Theorem 2.10. *In either context (1) or (2), we have a diagram:*

$$\mathcal{K} \begin{array}{c} \xleftarrow{i_*^*} \\ \xrightarrow{i_*} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j_!} \end{array} \mathcal{Q},$$

where:

- (1) *the canonical functor $j^! := \tilde{R}$ admits a left adjoint $j_!$;*
- (2) *the inclusion functor i_* admits a left adjoint i_*^* ;*
- (3) *The counit $i_*^* i_* \rightarrow 1_{\mathcal{K}}$, and the unit $1_{\mathcal{Q}} \rightarrow j^! j_!$ are isomorphisms;*
- (4) *There is a distinguished triangle of functors*

$$j_! j^! \rightarrow 1_{\mathcal{D}} \rightarrow i_*^* i_* \rightarrow;$$

- (5) *the (fully faithful) functor $j_!$ identifies \mathcal{Q} with the left orthogonal ${}^{\perp}\mathcal{K}$ to \mathcal{K} (moreover, ${}^{\perp}\mathcal{K}$ is the cocompletion of the essential image of L);*

Proof. To construct the left adjoint $j_!$, we take use the diagram 2 above, and set:

$$j_!(c) = \operatorname{colim} \left(Lc \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} LRLc \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} LRLRLc \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \dots \right)$$

The functor i_* is given as follows: first we have the functor

$$P := \operatorname{cone}(j_! j^! \rightarrow \operatorname{Id}_{\mathcal{D}}) : \mathcal{D} \rightarrow \mathcal{D}.$$

By construction, the essential image of P is precisely the subcategory \mathcal{K} . Thus we define $i_*^* : \mathcal{D} \rightarrow \mathcal{K}$ by $i_*^*(d) = P(d)$ (and $P \simeq i_*^* i_*$). \square

Remark 2.11. The category $\mathcal{Q} \simeq \mathcal{C}^T$ can be identified with the quotient \mathcal{D}/\mathcal{K} (either in the abelian or dg category setting). The inclusion $\mathcal{K} \rightarrow \mathcal{D}$ also has a right adjoint, $i^!$, by virtue of the fact that \mathcal{K} is closed under colimits. By taking cones as in the proof of Theorem 2.10, we obtain a right adjoint j_* to $j^!$. This situation is referred to as a *recollement*.

2.3. Monads and finite group actions. Let us keep the notation of Subsection 2.2. For simplicity, we work in the context (1) of abelian categories (that is the context which we will need in this paper). Suppose additionally that a finite group W acts on \mathcal{C} . This means that there are functors

$$w_* : \mathcal{C} \rightarrow \mathcal{C},$$

together with natural isomorphisms $\phi_{w,v} : w_* v_* \xrightarrow{\sim} (wv)_*$ satisfying the natural cocycle condition. There is an associated monad W_* acting on \mathcal{C} given by the formula:

$$W_*(c) = \bigoplus_{w \in W} w_*(c).$$

(In fact, the data of the functor W_* together with its monad structure is equivalent to the data of the action of W on \mathcal{C}). The category \mathcal{C}^W of W -equivariant objects, is equivalent to the category of modules for the monad W_* .

Remark 2.12. The functor W_* also admits the structure of a comonad acting on \mathcal{C} (and W -equivariant objects in \mathcal{C} can also be identified with comodules for this comonad). Considering this monad and comonad structure together, one arrives at the notion of Frobenius monad (we will not make use of the Frobenius structure in this paper).

Proposition 2.13. *Suppose there is an isomorphism of monads $T \simeq W_*$. Given an object $c \in \mathcal{C}^W$, W acts by automorphisms on the object $L(c)$ such that the object $j_!(c)$ is given by the coinvariants $L(c)_W$ of this action.*

Proof. The W action comes from the identities:

$$\mathrm{End}(L(c)) \simeq \mathrm{Hom}(c, RL(c)) \simeq \mathrm{Hom}(c, W_* c) \simeq \mathbb{Z}[W] \otimes \mathrm{End}(c).$$

The identification of $j_!(c)$ with the coinvariants is by inspection of the formula 2 for $j_!$ as a coequalizer. \square

Remark 2.14.

- (1) In the case when \mathcal{C} and \mathcal{D} are \mathbb{Q} -linear abelian categories, then the coinvariants of the W action are a direct summand. In particular, $j_!(c)$ is a direct summand of $L(c)$.
- (2) If \mathcal{C} and \mathcal{D} are \mathbb{C} -linear abelian categories and $c \in \mathcal{C}^T$ is a simple object, then $\mathrm{End}(L(c)) \simeq \mathbb{C}[W]$.

2.4. D -modules on Stacks. In this paper, all stacks appearing will be of the form Y/K where Y is a quasi-projective variety and K is an affine algebraic group (we will refer to such stacks as *quotient stacks*). The triangulated category of D -modules on such stacks was defined by Bernstein and Lunts [BL94] (in the language of sheaves) and Beilinson and Drinfeld [BD], and ∞ -categorical enhancements have been considered in [GR11] [BZN09]. Below we outline some of the key properties of the theory.

Let $X = Y/K$ be a smooth quotient stack. We will denote by $\mathbf{D}(X)$ or $\mathbf{D}(Y)^K$ the (unbounded) equivariant derived category of D -modules on X . This is a stable presentable ∞ -category which carries a t -structure. The heart of this t -structure is the abelian category of K -equivariant D -modules on Y , denote $\mathbf{M}(X) = \mathbf{M}(Y)^K$. The subcategory $\mathbf{D}_{coh}^b(X)$ consists of complexes with finitely many non-zero cohomology groups, each of which are coherent (as D -modules on Y). The category $\mathbf{D}(X)$ is compactly generated and the subcategory $\mathbf{D}_{com}^b(X)$ of compact objects is contained in $\mathbf{D}_{coh}^b(X)$ (these subcategories agree when X is a scheme, but may differ if X is a stack).

A *representable morphism of quotient stacks*, is a morphism that can be written in the form $f : U/K \rightarrow V/K$, corresponding to a K -equivariant morphism of varieties $\tilde{f} : U \rightarrow V$. We will need to work with a slightly more general class of morphisms than the representable ones.

A *very safe* morphism of quotient stacks shall mean a morphism of the form $f : U/K \rightarrow V/L$ where L is a quotient of K with unipotent kernel, and $U \rightarrow V$ is a K -equivariant morphism as before, where the action of K on V factors through L . The morphism f of stacks is smooth if and only if the corresponding morphism $\tilde{f} : U \rightarrow V$ is smooth; in that case, the *relative dimension of f* is given by the relative dimension of \tilde{f} minus the dimension of the kernel of $G \rightarrow H$.

Remark 2.15. The terminology *safe* comes from the paper [DG13] of Drinfeld-Gaitsgory. A very safe morphism of quotient stacks will be safe in the sense of that paper. All morphisms appearing in this paper will be very safe.

Example 2.16. Let B be a Borel subgroup of a reductive algebraic group, U the unipotent radical of B , and $H = B/U$. Then the morphism

$$B/_\text{ad} B \rightarrow H/_\text{ad} H$$

is very safe and smooth of relative dimension 0. Similarly, $U/_\text{ad} U$ is safe and smooth of relative dimension 0 over a point. On the other hand, the morphism $pt/H \rightarrow pt$ is not very safe (or indeed safe).

Given a very safe morphism of stacks $f : X \rightarrow Y$, we have functors:

$$f_* : \mathbf{D}(X) \rightarrow \mathbf{D}(Y),$$

and

$$f^! : {}^s\mathbf{D}(Y) \rightarrow \mathbf{D}(X).$$

Following the convention of [BZN09], we define $\mathbb{D}_X(\mathfrak{M})$ by the usual formula for D -module duality when $\mathfrak{M} \in \mathbf{D}_{\text{coh}}^b(X)$, and extend by continuity to define the functor

$$\mathbb{D}_X : \mathbf{D}(X) \rightarrow \mathbf{D}(X)'.$$

Remark 2.17. If $\mathfrak{M}, \mathfrak{N} \in \mathbf{D}_{\text{coh}}^b(X)$, then

$$R\text{Hom}(\mathfrak{M}, \mathfrak{N}) \simeq R\text{Hom}(\mathbb{D}(\mathfrak{N}), \mathbb{D}(\mathfrak{M})).$$

This formula fails in general, if we drop the coherence assumption.

Proposition 2.18.

- (1) If f is proper, then $f_* \simeq \mathbb{D}_Y f_* \mathbb{D}_X$ preserves coherence and is right adjoint to $f^!$. We sometimes write $f_!$ instead of f_* in that case.
- (2) If f is smooth of relative dimension d , then $f^!$ preserves coherence and $f^* := f^![-2d]$ is left adjoint to f_* . The functor $f^\dagger := f^![-d] = f^*[d]$ is t -exact, and $f^\dagger \simeq \mathbb{D}_X f^\dagger \mathbb{D}_Y$.
- (3) If

$$\begin{array}{ccc} X \times_W V & \xrightarrow{\tilde{f}} & V \\ \downarrow \tilde{f} & & \downarrow g \\ X & \xrightarrow{f} & W \end{array}$$

is a cartesian diagram of stacks, then the base change morphism is an isomorphism: $g^! f_* \cong \tilde{g}_* \tilde{f}^!$.

(4) We have the projection formula:

$$f_* (f^! \mathfrak{M} \otimes \mathfrak{N}) \simeq \mathfrak{M} \otimes f_*(\mathfrak{N}).$$

(5) The category $\mathbf{D}(X)$ carries a symmetric monoidal tensor product

$$\mathfrak{M} \otimes \mathfrak{N} := \Delta^!(\mathfrak{M} \boxtimes \mathfrak{N}) \simeq \mathfrak{M} \otimes_{\mathcal{O}_X} \mathfrak{N}[-\dim(X)].$$

(6) We have an internal Hom:

$$\mathcal{H}om(\mathfrak{M}, \mathfrak{N}) := \mathbb{D}(\mathfrak{M}) \otimes \mathfrak{N}.$$

If \mathfrak{M} and \mathfrak{N} are in $\mathbf{D}_{coh}^b(X)$ then

$$R\mathcal{H}om(\mathfrak{M}, \mathfrak{N}) = p_{X*} \mathcal{H}om(\mathfrak{M}, \mathfrak{N}).$$

Given quotient stacks X and Y , we have an equivalence:

$$\mathbf{D}(X \times Y) \xleftarrow{\sim} \mathrm{Fun}^L(\mathbf{D}(X), \mathbf{D}(Y)) \xrightarrow{\sim} \mathbf{D}(X) \otimes \mathbf{D}(Y).$$

$$\mathfrak{K} \longmapsto \Phi_{\mathfrak{K}} := q_Y (\mathfrak{K} \otimes q_X^!(-)) \quad .$$

where

$$X \xleftarrow{q_X} Y \xrightarrow{q_Y} Y,$$

are the projections. We refer to \mathfrak{K} as the kernel corresponding to the functor $\Phi_{\mathfrak{K}}$.

Note that if X is a stack with a stratification $Y = \bigsqcup_{i \in I} Y_i$ by locally closed substacks, then any object of $\mathbf{D}(Y)$ on Y is filtered by I . Moreover, if

$$X \xleftarrow{f} Y \xrightarrow{g} Z,$$

is a diagram of stacks, then the functor $f_* g^!$ is filtered by I : the functor $(f_* g^!)_J$ is given by $f_{J*} g_J^!$, where f_J (respectively g_J) is the restriction of f (respectively g) to $Y_J = \bigsqcup_{j \in J} Y_j$.

2.5. Fourier Transform and monodromic D -modules. Let V be a vector space with dual space V^* . Recall the Fourier transform functor defines t -exact inverse equivalences:

$$\mathbf{D}(V) \xrightleftharpoons[\mathbb{F}_{V^*}]{\mathbb{F}_V} \mathbf{D}(V^*)$$

Suppose $i : W \hookrightarrow V$ is the inclusion of a linear subspace; let $p : V^* \rightarrow W^*$ denote the adjoint to i . We have:

$$\mathbb{F}_W i^\dagger \simeq p_* \mathbb{F}_V : \mathbf{D}(V) \rightarrow \mathbf{D}(W^*)$$

$$\mathbb{F}_V i_* \simeq p^\dagger \mathbb{F}_W : \mathbf{D}(W) \rightarrow \mathbf{D}(V^*).$$

Now suppose A is a vector space acting on V as an additive group (e.g. A could be a subspace of V). Let $a : A \times V \rightarrow V$ denote the action map. A D -module $\mathfrak{M} \in \mathbf{D}(V)$ is called A -monodromic if the action of A on \mathfrak{M} (via the canonical map $A \rightarrow \mathrm{Vect}(V) \rightarrow \mathfrak{D}_V$) is locally finite. The full subcategory of monodromic D -modules on V will be denoted $\mathbf{D}^{A-mon}(V)$. We have an orthogonal decomposition

$$\mathbf{D}^{A-mon}(V) = \bigoplus_{\lambda \in A^*} \mathbf{D}_\lambda(V),$$

where $\mathbf{D}_\lambda(V)$ consists of objects for which A acts with generalized eigenvalues $\lambda \in A^*$. For each object \mathfrak{M} of $\mathbf{D}_\lambda(V)$, we have:

$$a^\dagger(\mathfrak{M}) \simeq \mathfrak{L}_\lambda \boxtimes \mathfrak{M},$$

where \mathfrak{L}_λ is the one dimensional flat connection on A corresponding to λ . Note that \mathfrak{M} is A -monodromic if and only if the support of $\mathbb{F}_V \mathfrak{M}$ intersects the orbits of A at finitely many points.

Lemma 2.19. *Let V be a vector space, C a conic subset of V and D a conic subset of V^* . Then the following are equivalent for an object $\mathfrak{N} \in \mathbf{D}(V)$:*

- (1) \mathfrak{M} is supported in C and $\mathbb{T}_V \mathfrak{M}$ is supported in D .
- (2) The singular support of \mathfrak{M} is contained in $C \times D \subseteq V \times V^* = T^*V$.

3. MACKEY THEORY

In this section we will define the functors of parabolic induction and restriction between categories of D -modules (or constructible sheaves), and define the Mackey filtration in that setting.

3.1. Conventions and Notation. Throughout this section, we will fix a connected reductive algebraic group G , and parabolic subgroups P and Q of G . We will denote by U and V the unipotent radicals of P and Q , and the Levi quotients will be denoted $L = P/U$, $M = Q/V$, respectively. The corresponding Lie algebras will be denoted by lower case fraktur letters as usual; thus the Lie algebras of G, P, Q, U, V, L, M shall be denoted $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}, \mathfrak{u}, \mathfrak{v}, \mathfrak{l}, \mathfrak{m}$ respectively.

Recall that an algebraic group acts on its Lie algebra by the adjoint action. For ease of reading we will denote the adjoint quotient stacks with an underline as follows: $\mathfrak{g}/G = \underline{\mathfrak{g}}, \mathfrak{p}/P = \underline{\mathfrak{p}}, \mathfrak{l}/L = \underline{\mathfrak{l}}$, etc. We have a diagram of stacks

$$(3) \quad \underline{\mathfrak{g}} \xleftarrow{r} \underline{\mathfrak{p}} \xrightarrow{s} \underline{\mathfrak{l}}.$$

Of course, there is an analogous diagram involving \mathfrak{q} and \mathfrak{m} . The fiber product $\underline{\mathfrak{q}} \times_{\underline{\mathfrak{g}}} \underline{\mathfrak{p}}$ will be denoted by ${}_{Q\text{st}}\underline{\mathfrak{p}}$ and referred to as the *Steinberg stack*. It is equipped with projections

$$\underline{\mathfrak{m}} \xleftarrow{\alpha} {}_{Q\text{st}}\underline{\mathfrak{p}} \xrightarrow{\beta} \underline{\mathfrak{l}}.$$

Explicitly we may write the Steinberg stack as a quotient

$$\left\{ (x, g) \in \mathfrak{g} \times G \mid x \in \mathfrak{q} \cap {}^{g^{-1}}\mathfrak{p} \right\} / (Q \times P),$$

where the $(q, p) \in Q \times P$ acts by sending (x, g) to $({}^q x, qgp^{-1})$. In this realization, the morphisms α and β are given by $\alpha(x, g) = (x + \mathfrak{v}) \in \mathfrak{m}$ and $\beta(x, g) = ({}^g x + \mathfrak{u}) \in \mathfrak{l}$. The Steinberg stack is stratified by the (finitely many) orbits of $Q \times P$ on G and all of the strata have the same dimension. For each orbit w in $Q \backslash G/P$, we denote by ${}_{Q\text{st}}\underline{\mathfrak{p}}^w$ the corresponding strata in ${}_{Q\text{st}}\underline{\mathfrak{p}}$.

Lemma 3.1. *Given any lift $\dot{w} \in G$ of w , we have an equivalence of stacks ${}_{Q\text{st}}\underline{\mathfrak{p}}^w \simeq (\mathfrak{q} \cap {}^{\dot{w}}\mathfrak{p})_{/ad} (Q \cap {}^{\dot{w}}P)$.*

Remark 3.2. The stack ${}_{Q\text{st}}\underline{\mathfrak{p}}$ is the bundle of Lie algebras associated to the inertia stack of $Q \backslash G/P$; the equidimensionality of the stratification may be seen as a consequence of the orbit stabilizer theorem.

Remark 3.3. The morphism $s : \underline{\mathfrak{p}} \rightarrow \underline{\mathfrak{l}}$ is not representable, however it is *very safe*, in the sense of Subsection 2.4. Note that s factorizes as

$$\underline{\mathfrak{p}} = \mathfrak{p}/P \rightarrow \mathfrak{l}/P \rightarrow \mathfrak{l}/L = \underline{\mathfrak{l}}.$$

The first morphism is representable, and the second morphism gives rise to a derived equivalence of D -modules (but the t -structure is shifted). The benefit of working with such non-representable morphisms is that the shifts that usually appear in the definitions of parabolic induction and restriction are naturally encoded in our definition.

To better understand Diagram 3 and the Steinberg stack, let us consider the following spaces. The P -flag variety, $\mathcal{F}\ell_{\mathfrak{p}} \simeq G/P$, is defined to be the collection of conjugates of \mathfrak{p} in \mathfrak{g} with its natural structure of a projective algebraic variety. The flag variety carries a tautological bundle of Lie algebras,

$$\tilde{\mathfrak{g}}_P = \{(x, \mathfrak{p}') \in G \times \mathcal{F}\ell_{\mathfrak{p}} \mid x \in \mathfrak{p}'\} \longrightarrow \mathcal{F}\ell_P.$$

The bundle of Lie algebras $\tilde{\mathfrak{g}}_P$ has a naturally defined ideal $\tilde{\mathfrak{g}}_U$ (whose fibers are conjugates of \mathfrak{u}) and corresponding quotient $\tilde{\mathfrak{g}}_L$. The group G acts on $\tilde{\mathfrak{g}}_P$ and $\tilde{\mathfrak{g}}_L$ and we have natural identifications $\tilde{\mathfrak{g}}_P/G = \mathfrak{p}/P = \underline{\mathfrak{p}}$ and $\tilde{\mathfrak{g}}_L/G = \mathfrak{l}/L = \underline{\mathfrak{l}}$.

Now we can reinterpret Diagram 3 as follows:

$$\begin{array}{ccccc} \mathfrak{g}/G & \xleftarrow{\rho} & \tilde{\mathfrak{g}}_P/G & \xrightarrow{\sigma'} & \tilde{\mathfrak{g}}_L/G \\ \downarrow \wr & & \downarrow \wr & & \downarrow \sigma'' \\ \underline{\mathfrak{g}} & \xleftarrow{r} & \underline{\mathfrak{p}} & \xrightarrow{s} & \underline{\mathfrak{l}} \end{array}$$

The morphism σ' is representable and smooth of relative dimension $\dim(\mathfrak{u})$, whereas σ'' is smooth of relative dimension $-\dim(\mathfrak{u})$. Thus the morphism s is smooth of relative dimension 0. Note that σ'' induces an equivalence on the category of D -modules, which preserves the t -structures up to a shift of $\dim(\mathfrak{u})$. Thus the top row of the diagram induces the same functor on D -modules up to a shift.

The Steinberg stack may be written in these terms as follows:

$${}_{Q}\underline{\mathbf{st}}_P = (\tilde{\mathfrak{g}}_Q \times_{\mathfrak{g}} \tilde{\mathfrak{g}}_P)/G = \{(x, \mathfrak{q}', \mathfrak{p}') \in \mathfrak{g} \times \mathcal{F}\ell_Q \times \mathcal{F}\ell_P \mid x \in \mathfrak{p}' \cap \mathfrak{q}'\}/G.$$

The stratification on ${}_{Q}\underline{\mathbf{st}}_P$ is identified in these terms of the relative position of Q' and P' .

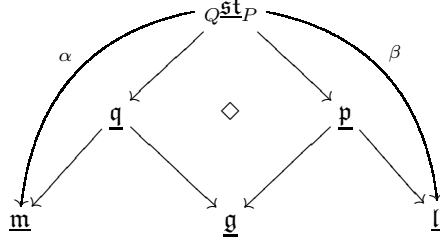
3.2. Induction, Restriction, and Intertwining Functors. The functors of parabolic induction and restriction are defined by:

$$r_* s^! = \mathbf{Ind}_{P,L}^G : \mathbf{D}(\underline{\mathfrak{l}}) \rightleftarrows \mathbf{D}(\underline{\mathfrak{g}}) : \mathbf{Res}_{P,L}^G = s_* r^!$$

where the notation is as in Diagram 3 above. The morphism r is proper, and s is smooth of relative dimension 0, thus $\mathbf{Ind}_{P,L}^G$ is left adjoint to $\mathbf{Res}_{P,L}^G$. We define the functor

$$\mathbf{St} = {}_{M,Q}\mathbf{St}_{P,L}^G := \mathbf{Res}_{Q,M}^G \mathbf{Ind}_{P,L}^G : \mathbf{D}(\underline{\mathfrak{l}}) \rightarrow \mathbf{D}(\underline{\mathfrak{m}}),$$

(we will often drop the subscripts and superscripts when the context is clear). By base change, we have $\mathbf{St} \simeq \alpha_* \beta^!$, where:



Recall that the stack $Q\mathbf{st}_P$ has a stratification indexed by double cosets $Q \backslash G / P$, with strata $Q\mathbf{st}_P^w$. Thus the functor \mathbf{St} has a filtration (in the sense of Definition 2.5) indexed by the poset $Q \backslash G / P$ (we will refer to this filtration as the Mackey filtration).

3.3. The Mackey Strata. For each $g \in G$, consider the conjugate parabolic gP in G and its Levi quotient gL . The image of $Q \cap {}^gP$ in M is a parabolic subgroup of M which we will denote $M \cap {}^gP$. The corresponding Levi quotient will be denoted $M \cap {}^gL$. Similarly, the image of $Q \cap {}^gP$ in gL is a parabolic subgroup, denoted $Q \cap {}^gP$, whose Levi subgroup is canonically identified with $M \cap {}^gL$. Analogous notation will be adopted for the corresponding Lie algebras.

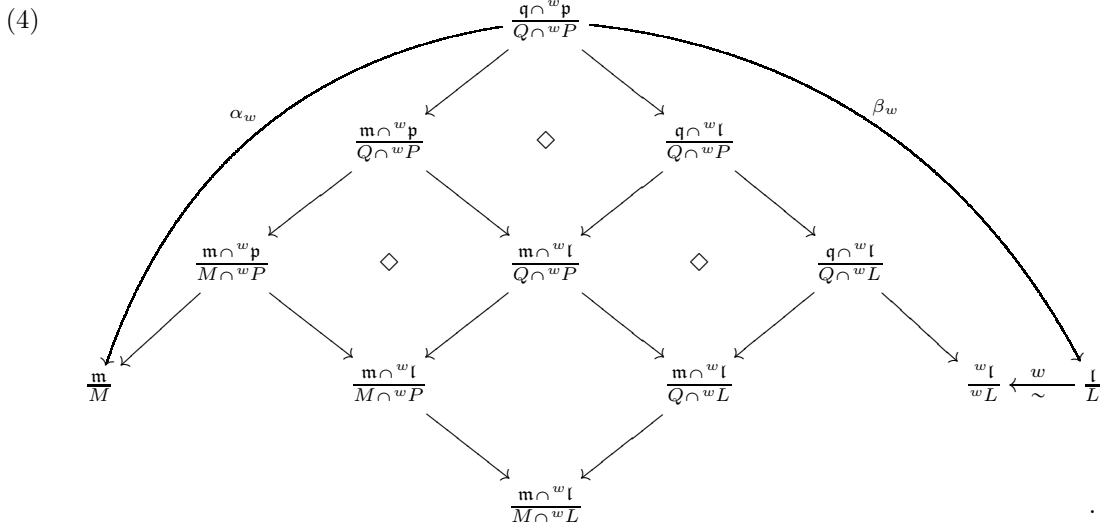
Remark 3.4. If we choose Levi splittings of L in P and M in Q such that $M \cap {}^gL$ contains a maximal torus of G , then the notation above may be interpreted literally.

The conjugation morphism ${}^g(-) : \mathbf{l} \rightarrow {}^g\mathbf{l}$ gives rise to an equivalence $g_* : \mathbf{D}(\mathbf{l}) \rightarrow \mathbf{D}({}^g\mathbf{l})$.

Proposition 3.5 (Mackey Filtration). *For each lift \dot{w} of w , there is an equivalence*

$$\mathbf{St}^w \simeq \mathrm{Ind}_{M \cap {}^{\dot{w}}P, M \cap {}^{\dot{w}}L}^M \mathrm{Res}_{Q \cap {}^{\dot{w}}L, M \cap {}^{\dot{w}}L}^{{}^{\dot{w}}L} \dot{w}_* : \mathbf{D}(\mathbf{l}) \rightarrow \mathbf{D}(\underline{\mathbf{m}}).$$

Proof. Let us pick a lift $\dot{w} \in G$ of w . Recall the isomorphism: $Q\mathbf{st}_P^w \simeq (\mathbf{q} \cap {}^w\mathbf{p}) / {}_{ad}(Q \cap {}^wP)$. Consider the commutative diagram:



Traversing the correspondence along the top of the diagram (from right to left) gives the functor \mathbf{St}_w , whereas traversing the two correspondences along the very bottom of the diagram gives the composite $\mathbf{Ind}_{M \cap {}^w P, M \cap {}^w L}^M \circ \mathbf{Res}_{Q \cap {}^w L, M \cap {}^w L}^{{}^w L} \circ {}^w \dot{*}$. The three squares marked \diamond are all cartesian. The lowest square is not cartesian, however it is still true that the two functors $\mathbf{D}(\mathfrak{m} \cap {}^w \mathfrak{l})^{Q \cap {}^w L} \rightarrow \mathbf{D}(\mathfrak{m} \cap {}^w \mathfrak{l})^{M \cap {}^w P}$ obtained by either traversing the top or bottom of this square are naturally isomorphic (the base-change morphism is an isomorphism). Indeed, the stacks only differ by trivial actions of unipotent groups, thus all the D -module functors in the square are equivalences, differing at worst by a cohomological shift. The fact that the shift is trivial follows just from the commutativity of the diagram. Thus the entire diagram behaves as if it were a base change diagram as required. \square

4. PROPERTIES OF INDUCTION AND RESTRICTION

In this section we will keep the conventions of Section 3 (so P and Q are parabolic subgroups of G with Levi quotients L , M etc.), but additionally fix Levi splittings of L and M inside P and Q . Consider the set $\mathcal{S}(M, L)$ which is defined to be the set of conjugates, ${}^g L$ of L such that $M \cap {}^g L$ contains a maximum torus of G (for convenience, let us assume that $M \cap L$ contains a maximum torus H of G). There is a bijection $M \backslash \mathcal{S}(M, L) / L \simeq Q \backslash G / P$.

4.1. The regular locus. An element $x \in \mathfrak{l}$ is called *regular* if $C_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [x, y] = 0\} \subseteq \mathfrak{l}$. The set of regular elements in \mathfrak{l} is an open subset of denoted $\mathfrak{l}^{\text{reg}}$. We also define $\mathfrak{p}^{\text{reg}} := \mathfrak{l}^{\text{reg}} + \mathfrak{u}$.

The open subset $\mathfrak{g}_{\leq(L)}$ is defined to be the image ${}^G \mathfrak{l}^{\text{reg}}$ of $\mathfrak{l}^{\text{reg}}$ under the G -action. Similarly, we define $\mathfrak{g}_{<(L)}$ to be the union of $\mathfrak{g}_{(L')}$ as L' ranges over proper Levi subgroups of L .

The loci in the Lie algebras defined above are preserved by the adjoint actions of the corresponding algebraic group, and we will denote the adjoint quotients by $\underline{\mathfrak{l}}^{\text{reg}}$, $\underline{\mathfrak{p}}^{\text{reg}}$, and $\underline{\mathfrak{g}}_{\leq(L)}$ respectively (in fact $\underline{\mathfrak{l}}^{\text{reg}} \simeq \underline{\mathfrak{p}}^{\text{reg}}$ as we will see in the following proposition).

Remark 4.1. Let us consider the case when $P = B$ is a Borel subgroup, and $L = H$ is a maximal torus. Then the regular locus $\mathfrak{h}^{\text{reg}}$ is the complement of the root hyperplanes. The sub $\mathfrak{g}_{\leq(H)} = \mathfrak{g}^{rs}$ is the locus of regular semisimple elements in \mathfrak{g} .

Let $c : \underline{\mathfrak{l}} \rightarrow \underline{\mathfrak{p}}$ and $d : \underline{\mathfrak{l}} \rightarrow \underline{\mathfrak{g}}$ denote the morphisms corresponding to the inclusions $\mathfrak{l} \hookrightarrow \mathfrak{p}$ and $\mathfrak{l} \hookrightarrow \mathfrak{g}$. We have a diagram:

$$(5) \quad \begin{array}{ccc} & \underline{\mathfrak{p}} & \\ r \swarrow & \curvearrowright c & \searrow s \\ \underline{\mathfrak{g}} & \xleftarrow{d} & \underline{\mathfrak{l}} \end{array}$$

where $rc = d$.

Proposition 4.2. [Lusztig [Lus84]] Diagram 5 above restricts to a diagram

$$\begin{array}{ccc} & \underline{\mathfrak{p}}^{\text{reg}} & \\ r^{\text{reg}} \swarrow & \curvearrowright c^{\text{reg}} & \searrow s^{\text{reg}} \\ \underline{\mathfrak{g}}_{\leq(L)} & \xleftarrow{d^{\text{reg}}} & \underline{\mathfrak{l}}^{\text{reg}} \end{array}$$

where:

- (1) The morphisms c^{reg} and d^{reg} define inverse isomorphisms between \mathbf{p}^{reg} and \mathbf{l}^{reg} ,
- (2) The morphism r^{reg} (equivalently d^{reg}) is étale of degree $|W(G, H)/\bar{W}(L, H)|$.

According to Proposition 4.2 parabolic induction and restriction on the regular loci are just given by pullback and pushforward along the étale map $d^{\text{reg}} : \mathbf{l}^{\text{reg}} \rightarrow \mathbf{g}_{\leq(L)}$. In particular, they are independent of the choice of parabolic subgroup P containing L .

Proposition 4.3. *Suppose $\mathfrak{M} \in \mathbf{D}(\mathbf{g})$ and $\mathbf{Res}_{P,L}^G(\mathfrak{M}) \simeq 0$. Then $\mathfrak{M}|_{\mathbf{g}_{\leq(L)}} \simeq 0$.*

Proof. Given \mathfrak{M} with $\mathbf{Res}_{P,L}^G(\mathfrak{M}) \simeq 0$, we have

$$d^!(\mathfrak{M}) \simeq \mathbf{Res}_{P,L}^G(\mathfrak{M})|_{\mathbf{l}^{\text{reg}}} \simeq 0.$$

But d is étale, so $d^!$ is conservative, and thus $\mathfrak{M}|_{\mathbf{g}_{\leq(L)}} \simeq 0$ as required. \square

The next result explains how the Mackey filtration is naturally split on the regular locus. Recall that we have fixed another Levi subgroup M inside a parabolic Q (in addition to L and P).

Proposition 4.4. *Let $j : \mathbf{m}^{\text{reg}} \hookrightarrow \mathbf{m}$ denote the inclusion. There is a canonical natural isomorphism:*

$$j^! \circ {}_{M,Q}\mathbf{St}_{P,L}^w \simeq \bigoplus_{w \in M \backslash \mathcal{S}(M,L)/L} j^! \circ {}_{M,Q}\mathbf{St}_{P,L}^w.$$

Proof. We consider the open substack ${}_{Q}\mathbf{st}_P^{\text{reg}} := \mathbf{q}^{\text{reg}} \times_{\mathbf{g}} \mathbf{p}$ of ${}_{Q}\mathbf{st}_P$. We denote by ${}_{Q}\mathbf{st}_P^w$ the intersection of ${}_{Q}\mathbf{st}_P^w$ with ${}_{Q}\mathbf{st}_P^{\text{reg}}$. Recall that the functor ${}_{M,Q}\mathbf{St}_{P,L}^w$ is given by $\alpha_*\beta^!$ where:

$$\mathbf{m} \xleftarrow{\alpha} {}_{Q}\mathbf{st}_P \xrightarrow{\beta} \mathbf{l}$$

Thus $j^! {}_{M,Q}\mathbf{St}_{P,L}^w$ is given by $\alpha_*^{\text{reg}}\beta^!$ (where $\alpha^{\text{reg}} : {}_{Q}\mathbf{st}_P^{\text{reg}} \rightarrow \mathbf{m}^{\text{reg}}$). The next lemma gives the result. \square

Lemma 4.5. *Each stratum in the stratification*

$${}_{Q}\mathbf{st}_P^{\text{reg}} \simeq \bigsqcup_{w \in Q \backslash G/P} {}_{Q}\mathbf{st}_P^w.$$

is both open and closed (i.e. the stratification is a disjoint union of stacks).

Proof. Consider the opposite parabolic \overline{Q} of Q with respect to the Levi subgroup M (with Lie algebra $\overline{\mathbf{q}}$ etc.). We have isomorphisms $\mathbf{q}^{\text{reg}} \simeq \mathbf{m}^{\text{reg}} \simeq \overline{\mathbf{q}}^{\text{reg}}$, and thus

$${}_{Q}\mathbf{st}_P^{\text{reg}} \simeq \mathbf{m}^{\text{reg}} \times_{\mathbf{g}} \mathbf{p} \simeq \overline{{}_{Q}\mathbf{st}}_{P^{\text{reg}}}.$$

Note that the bijections

$$Q \backslash G/P \simeq M \backslash \mathcal{S}(M,L)/L \simeq \overline{Q} \backslash G/P$$

induce the opposite partial order on $M \backslash \mathcal{S}(M,L)/L$. Thus the closure relations amongst the strata ${}_{Q}\mathbf{st}_P^w$ are self opposed. It follows that each stratum is both open and closed, as required. \square

4.2. The Lusztig Stratification and Galois Covers. Let us define the closed subvariety \mathfrak{l}_\heartsuit of \mathfrak{l} by $\mathcal{N}_L + \mathfrak{z}(\mathfrak{l})$. Similarly, we write

$$\mathfrak{l}_\heartsuit^{\text{reg}} := \mathfrak{l}_\heartsuit \cap \mathfrak{l}^{\text{reg}} = \mathcal{N}_L + \mathfrak{z}(\mathfrak{l})^{\text{reg}}.$$

We define $\mathfrak{g}_{(L)}$ to be ${}^G\mathfrak{l}_\heartsuit^{\text{reg}}$. As usual, we denote $\mathfrak{l}_\heartsuit/L$ by \mathfrak{l}_\heartsuit , $\mathfrak{g}_{(L)}/G$ by $\underline{\mathfrak{g}}_{(L)}$, etc. Note that $\underline{\mathfrak{g}}_{(L)}$ is a closed substack of $\underline{\mathfrak{g}}_{\leq(L)}$, and $\underline{\mathfrak{g}}_{\leq(L)}$ is open in $\underline{\mathfrak{g}}$.

Proposition 4.6 (Lusztig). *The substacks $\underline{\mathfrak{g}}_{(L)}$ form a locally closed partition of $\underline{\mathfrak{g}}$ indexed by the poset of conjugacy classes of Levi subgroups of G .*

In fact, Lusztig described a refinement of the partition $\underline{\mathfrak{g}}_{(L)}$ to a stratification as follows. For each nilpotent orbit \mathcal{O} in \mathcal{N}_L , we write $\mathfrak{l}^{\mathcal{O},\text{reg}} = \mathcal{O} + \mathfrak{z}(\mathfrak{l})^{\text{reg}}$, and $\underline{\mathfrak{g}}_{(L,\mathcal{O})} = {}^G\mathfrak{l}^{\mathcal{O},\text{reg}}$. The substack $\underline{\mathfrak{g}}_{(L)}$ is stratified by $\underline{\mathfrak{g}}_{(L,\mathcal{O})}$ as \mathcal{O} range over G -conjugacy classes of nilpotent orbits in L .

Note that, for each Levi subgroup L , the open sets $\underline{\mathfrak{g}}_{\leq(L)}$ are the union of $\underline{\mathfrak{g}}_{(M)}$ for Levi subgroups M of L , so the notation makes sense. Similarly $\underline{\mathfrak{g}}_{\geq(L)}$ is the union of $\underline{\mathfrak{g}}_{(M)}$ for all $(M) \geq (L)$; it is also the closure of $\underline{\mathfrak{g}}_{(L)}$ in $\underline{\mathfrak{g}}$.

We have the following refinement of Proposition 4.2. Recall that $W(G, L) = N_G(L)/L$ denotes the relative Weyl group.

Proposition 4.7 (Lusztig). *The Grothendieck-Springer diagram restricts to:*

$$(6) \quad \begin{array}{ccccc} \underline{\mathfrak{g}}_{\geq(L)} & \xleftarrow{r^\heartsuit} & \underline{\mathfrak{p}}^\heartsuit & \xrightarrow{s^\heartsuit} & \underline{\mathfrak{l}}^\heartsuit \\ \uparrow & & \uparrow & & \uparrow \\ \underline{\mathfrak{g}}_{(L)} & \xleftarrow{r^{\heartsuit,\text{reg}}} & \underline{\mathfrak{p}}^{\heartsuit,\text{reg}} & \xrightarrow{s^{\heartsuit,\text{reg}}} & \underline{\mathfrak{l}}^{\heartsuit,\text{reg}} \end{array}$$

where

- (1) The vertical arrows are dense open embeddings,
- (2) The morphism $s^{\heartsuit,\text{reg}}$ is an isomorphism, and $r^{\heartsuit,\text{reg}}$ is a $W(L, G)$ -Galois cover.
- (3) The morphism r^\heartsuit is small.

In other words, the morphism of stacks $\mathfrak{l}/W(G, L) \rightarrow \underline{\mathfrak{g}}$ becomes an isomorphism upon restricting to $\mathfrak{l}^{\heartsuit,\text{reg}}$. In particular:

Corollary 4.8. *There are commutative diagrams:*

$$\begin{array}{ccc} \mathbf{D}(\underline{\mathfrak{g}}_{(L)}) & \begin{array}{c} \xrightarrow{\text{reg Res}_L^G} \\ \xleftarrow{\text{reg Ind}_L^G} \end{array} & \mathbf{D}(\mathfrak{l}^{\heartsuit,\text{reg}}) \\ \uparrow \wr & & \uparrow \text{II} \\ \mathbf{D}(\mathfrak{l}^{\heartsuit,\text{reg}})^{W(L)} & \begin{array}{c} \xrightarrow{\Upsilon_W} \\ \xleftarrow{\Gamma^W} \end{array} & \mathbf{D}(\mathfrak{l}^{\heartsuit,\text{reg}}), \end{array}$$

where Υ_W and Γ^W are the forgetful, and equivariantization functors respectively.

4.3. Fourier Transform. Let us choose a G -invariant form on \mathfrak{g} to identify \mathfrak{g} with \mathfrak{g}^* (this is just convenience; the choice of form will not be relevant). The Fourier transform functor for D -modules is then an involution:

$$\mathbb{F}_{\mathfrak{g}} : \mathbf{D}(\mathfrak{g}) \xrightarrow{\sim} \mathbf{D}(\mathfrak{g}).$$

Lemma 4.9 (Lusztig (see also Mirkovic [Mir04], 4.2)). *The functors of induction and restriction commute with the Fourier transform functor for all parabolic subgroups P with Levi factor L :*

$$\begin{aligned} \mathbb{F}_{\mathfrak{l}} \mathbf{Res}_{P,L}^G &\simeq \mathbf{Res}_{P,L}^G \mathbb{F}_{\mathfrak{g}} \\ \mathbb{F}_{\mathfrak{g}} \mathbf{Ind}_{P,L}^G &\simeq \mathbf{Ind}_{P,L}^G \mathbb{F}_{\mathfrak{l}}. \end{aligned}$$

The following lemma will be used multiple times throughout the remainder of the paper.

Lemma 4.10. *Suppose L is a proper Levi subgroup of G and $\mathfrak{N} \in \mathbb{D}(\mathfrak{l})$ has the property that*

$$\mathfrak{N}|_{\mathfrak{l}^{\text{reg}}} \simeq 0 \simeq (\mathbb{F}_{\mathfrak{l}} \mathfrak{N})|_{\mathfrak{l}^{\text{reg}}}.$$

Then $\mathfrak{N} \simeq 0$.

Example 4.11. If $G = SL_2$, and $L \simeq \mathbb{C}^\times$, the maximal torus, then Lemma 4.10 says that if a D -module $\mathfrak{N} \in \mathbf{D}(\mathbb{C})$ and its Fourier transform are both supported at 0, then $\mathfrak{N} \simeq 0$. This statement is a direct consequence of “Bernstein’s inequality”. Lemma 4.10 is a generalization of this idea.

Proof of Lemma 4.10. Note that $\mathfrak{z}(\mathfrak{l}) \neq 0$ as L is a proper Levi. Pick a line $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{l})$ such that $\mathfrak{a} \cap (\mathfrak{z}(\mathfrak{l}) \setminus \mathfrak{z}(\mathfrak{l})^{\text{reg}}) = \{0\}$ (this condition holds generically). The center $\mathfrak{z}(\mathfrak{l})$ and thus \mathfrak{a} acts on \mathfrak{l} as an additive group. On the one hand,

$$(7) \quad \mathfrak{N}|_{\mathfrak{l}^{\text{reg}}} \simeq 0.$$

means that the \mathfrak{a} orbits intersect the support of \mathfrak{N} at finitely many points. Thus the Fourier transform $\mathbb{F}_{\mathfrak{l}} \mathfrak{N}$ is monodromic for the action of \mathfrak{a} . On the other hand,

$$(8) \quad (\mathbb{F}_{\mathfrak{l}} \mathfrak{N})|_{\mathfrak{l}^{\text{reg}}} \simeq 0.$$

which says that the orbits of \mathfrak{a} intersect the support of $\mathbb{F}_{\mathfrak{l}} \mathfrak{N}$ at finitely many points. The only way this can occur is if $\mathfrak{N} \simeq 0$ as required. \square

4.4. Preservation of Coherence and t -structure. The goal of this subsection is to prove that functors of parabolic induction and restriction preserve the t -structure and bounded coherent complexes.

Proposition 4.12. *The functor $\mathbf{Res}_{P,L}^G$ preserve the t -structure.*

Proof. Suppose $\mathfrak{M} \in \mathbf{D}(\mathfrak{g})^{\geq 0}$. Consider the truncation $\tau^{\leq -1} \mathbf{Res}_{P,L}^G$; we wish to show that it vanishes. We have

$$\tau^{\leq -1} \mathbf{Res}_{P,L}^G(\mathfrak{M})|_{\mathfrak{l}^{\text{reg}}} \simeq \tau^{\leq -1} d^{\text{reg}!}(\mathfrak{M}|_{\mathfrak{l}^{\text{reg}}}) \simeq 0,$$

as d^{reg} is an étale map and thus $d^{\text{reg}!}$ preserves t -structure. Thus $\tau^{\leq -1} \mathbf{Res}_{P,L}^G(\mathfrak{M})$ is supported on $\mathfrak{l} - \mathfrak{l}^{\text{reg}}$. In the same way, we have $\mathbf{Res}_{P,L}^G(\mathbb{F}_{\mathfrak{g}} \mathfrak{M}) \simeq \mathbb{F}_{\mathfrak{l}} \mathbf{Res}_{P,L}^G(\mathfrak{M}) \simeq 0$ (the first isomorphism uses Lemma 4.9). Thus, by Lemma 4.10, we have that $\tau^{\leq -1} \mathbf{Res}_{P,L}^G(\mathfrak{M}) \simeq 0$ as required. The same argument applies when $\mathfrak{M} \in \mathbf{D}(\mathfrak{l})^{\leq 0}$, replacing $\tau^{\leq -1}$ with $\tau^{\geq 1}$. Thus $\mathbf{Res}_{P,L}^G$ is t -exact, as required. \square

It follows from Proposition 4.12 that we have a well defined functor

$$\mathbf{res}_{P,L}^G : \mathbf{M}(\underline{\mathfrak{g}}) \rightarrow \mathbf{M}(\underline{\mathfrak{l}}).$$

Proposition 4.13. *Suppose $\mathfrak{M} \in \mathbf{M}(\underline{\mathfrak{g}})$ is coherent. Then $\mathbf{res}_{P,L}^G(\mathfrak{M})$ is coherent.*

Proof. Let $\mathfrak{N} = \mathbf{res}_{P,L}^G(\mathfrak{M})$. To show it is coherent, it is enough to consider the underlying $\mathfrak{D}_{\mathfrak{l}}$ -module of \mathfrak{N} (i.e. we can forget the equivariant structure). Recall that $\mathfrak{N}|_{\mathfrak{l}^{\text{reg}}} \simeq d^{\text{reg}!}\mathfrak{M}$, where $d^{\text{reg}} : \mathfrak{l}^{\text{reg}} \rightarrow \underline{\mathfrak{g}}$ is étale. Thus $\mathfrak{N}|_{\mathfrak{l}^{\text{reg}}}$ is coherent. According to the Coherent Extension Lemma ([HTT08] Corollary 1.4.17), there is a coherent $\mathfrak{D}_{\mathfrak{l}}$ -submodule \mathfrak{N}_1 such that $\mathfrak{N}_1|_{\mathfrak{l}^{\text{reg}}} = \mathfrak{N}|_{\mathfrak{l}^{\text{reg}}}$. The same is true for the Fourier transform $\mathbb{F}_{\mathfrak{l}}\mathfrak{N}$, thus we have a coherent $\mathfrak{D}_{\mathfrak{l}}$ -submodule of $\mathbb{F}_{\mathfrak{l}}\mathfrak{N}$ which such that $\mathfrak{N}_2|_{\mathfrak{l}^{\text{reg}}} = \mathbb{F}_{\mathfrak{l}}\mathfrak{N}|_{\mathfrak{l}^{\text{reg}}}$. We consider the coherent $\mathfrak{D}_{\mathfrak{l}}$ -submodule \mathfrak{N}' of \mathfrak{N} generated by \mathfrak{N}_1 and $\mathbb{F}_{\mathfrak{l}}\mathfrak{N}_2$. We wish to show that $\mathfrak{N}' = \mathfrak{N}$. To see this, note that the quotient $\mathfrak{N}/\mathfrak{N}'$ is a quotient of $\mathfrak{N}/\mathfrak{N}_1$, and thus is supported on the complement of $\mathfrak{l}^{\text{reg}}$. Similarly, $\mathbb{F}_{\mathfrak{l}}(\mathfrak{N}/\mathfrak{N}')$ is a quotient of $\mathbb{F}_{\mathfrak{l}}(\mathfrak{N})/\mathfrak{N}_2$ and thus is also supported on the complement of $\mathfrak{l}^{\text{reg}}$. By Lemma 4.10, $\mathfrak{N}/\mathfrak{N}' = 0$, so $\mathfrak{N} = \mathfrak{N}'$ is coherent as required. \square

It follows immediately from Proposition 4.13 that the functor $\mathbf{Res}_{P,L}^G$ preserves the subcategory of bounded coherent complexes $\mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{g}})$. The same is true automatically for $\mathbf{Ind}_{P,L}^G$ as parabolic induction is a composite of a smooth pullback and proper pushforward. Thus we have:

Proposition 4.14. *Parabolic induction and restriction restrict to an adjunction:*

$$\mathbf{Ind}_{P,L}^G : \mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{l}}) \xrightleftharpoons{\quad} \mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{g}}) : \mathbf{Res}_{P,L}^G$$

4.5. Second adjunction. Proposition 4.13 means that the composite functor $\mathbb{D}_{\mathfrak{l}}\mathbf{Res}_{P,L}^G\mathbb{D}_{\underline{\mathfrak{g}}}$ is a well defined exact functor $\mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{g}}) \rightarrow \mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{l}})$, which is left adjoint to $\mathbb{D}_{\underline{\mathfrak{g}}}\mathbf{Ind}_{P,L}^G\mathbb{D}_{\underline{\mathfrak{l}}} \simeq \mathbf{Ind}_{P,L}^G$. Extending by continuity, we have an adjunction:

$$\mathbf{Ind}_{P,L}^G : \mathbf{D}(\underline{\mathfrak{l}}) \xrightleftharpoons{\quad} \mathbf{D}(\underline{\mathfrak{g}}) : \overline{\mathbf{Res}}_{P,L}^G := \mathbb{D}_{\mathfrak{l}}\mathbf{Res}_{P,L}^G\mathbb{D}_{\underline{\mathfrak{g}}}$$

Note that there is a natural isomorphism $\mathbf{Res}_{P,L}^G\mathfrak{M}|_{\mathfrak{l}^{\text{reg}}} \simeq \overline{\mathbf{Res}}_{P,L}^G\mathfrak{M}|_{\mathfrak{l}^{\text{reg}}}$. The same argument as Proposition 4.12 gives the following:

Lemma 4.15. *The functor $\overline{\mathbf{Res}}_{P,L}^G$ is t -exact.*

Now, as $\mathbf{Ind}_{P,L}^G$ is left adjoint to $\mathbf{Res}_{P,L}^G$, it follows from the axioms of t -structures that

$$\mathbf{Ind}_{P,L}^G(\mathbf{D}(\underline{\mathfrak{l}})^{\leq 0}) \subseteq \mathbf{D}(\underline{\mathfrak{g}})^{\leq 0}.$$

Similarly, as $\mathbf{Ind}_{P,L}^G$ is right adjoint to $\overline{\mathbf{Res}}_{P,L}^G$, we have

$$\mathbf{Ind}_{P,L}^G(\mathbf{D}(\underline{\mathfrak{l}})^{\geq 0}) \subseteq \mathbf{D}(\underline{\mathfrak{g}})^{\geq 0}.$$

Thus we have that $\mathbf{Ind}_{P,L}^G$ is t -exact. Thus we obtain:

Proposition 4.16. *There are adjoint functors:*

$$\mathbf{ind}_{P,L}^G : \mathbf{M}(\underline{\mathfrak{l}}) \xrightleftharpoons{\quad} \mathbf{M}(\underline{\mathfrak{g}}) : \overline{\mathbf{res}}_{P,L}^G$$

The following result will not be needed in the remainder of the paper; we include it for completeness. It can be proved using Braden's hyperbolic localization technique [Bra03], see [DG14].

Theorem 4.17. *There is a natural isomorphism $\overline{\mathbf{Res}}_{P,L}^G \simeq \mathbf{Res}_{\overline{P},L}^G$, where \overline{P} is the opposite parabolic to P with respect to a splitting of L in P .*

Remark 4.18. Note that we have a “cycle of adjoints” of length 4:

$$\dots \dashv \mathbf{Ind}_{P,L}^G \dashv \mathbf{Res}_{P,L}^G \dashv \mathbf{Ind}_{\overline{P},L}^G \dashv \mathbf{Res}_{\overline{P},L}^G \dashv \mathbf{Ind}_{P,L}^G \dashv \dots$$

5. DECOMPOSITION ACCORDING TO LEVI SUBGROUPS

The goal of this section is to define, for each Levi subgroup L of G , subcategories $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ of $\mathbf{D}(\mathfrak{g})$ consisting of objects generated by parabolic induction from L . In fact, it will be more natural to consider the complementary subcategories $\mathbf{D}(\mathfrak{g})_{\nless(L)}$ first; these consist of objects which are killed by parabolic restriction to L . The first goal will be to show that the condition of being killed by parabolic restriction to L is independent of the choice of parabolic containing L . This will be achieved by giving another characterization of $\mathbf{D}(\mathfrak{g})_{\nless(L)}$ in terms of the (singular) support.

5.1. The kernel of parabolic restriction.

Proposition 5.1. *Given an object $\mathfrak{M} \in \mathbf{D}(\mathfrak{g})$, the following are equivalent:*

- (1) $\mathbf{Res}_{P,L}^G(\mathfrak{M}) \simeq 0$ for any parabolic subgroup P containing L as a Levi factor.
- (2) $\mathbf{Res}_{P,L}^G(\mathfrak{M}) \simeq 0$ for some parabolic subgroup P containing L as a Levi factor.
- (3) Both \mathfrak{M} and $\mathbb{T}_{\mathfrak{g}}\mathfrak{M}$ are supported on the closed substack $\mathfrak{g}_{\nless(L)}$.
- (4) The singular support of \mathfrak{M} is contained in the closed substack

$$\{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid (x, y) \in \mathfrak{g}_{\nless(L)} \times \mathfrak{g}_{\nless(L)}, [x, y] = 0\} / G \subseteq T^*\mathfrak{g}.$$

Proof. Tautologically (1) \Rightarrow (2). Let us show (2) \Rightarrow (3). Given \mathfrak{M} as in (2), we have that \mathfrak{M} is supported in $\mathfrak{g}_{\nless(L)}$ by Proposition 4.3. By Lemma 4.9, the Fourier transform $\mathbb{T}_{\mathfrak{g}}(\mathfrak{M})$ also satisfies the conditions of (2), and thus $\mathbb{T}_{\mathfrak{g}}(\mathfrak{M})$ is also supported on $\mathfrak{g}_{\nless(L)}$ as required.

Now let us show (3) \Rightarrow (1). Given \mathfrak{M} as in (3), pick a parabolic P containing L as a Levi factor and consider $\mathfrak{N} := \mathbf{Res}_{P,L}^G(\mathfrak{M})$. Then

$$(9) \quad \mathfrak{N}|_{\mathfrak{I}^{\text{reg}}} \simeq \mathbf{Res}_{P,L}^G(\mathfrak{M}|_{\mathfrak{g}_{\leq(L)}}) \simeq 0.$$

Similarly

$$(10) \quad (\mathbb{T}_{\mathfrak{g}}\mathfrak{N})|_{\mathfrak{I}^{\text{reg}}} \simeq \mathbf{Res}_{P,L}^G(\mathbb{T}_{\mathfrak{g}}\mathfrak{M}|_{\mathfrak{g}_{\leq(L)}}) \simeq 0.$$

Thus the result follows from Lemma 4.10.

It remains to show the equivalence of (3) and (4). This follows from Lemma 2.19. Note that condition (4) will not be used for the remainder of this paper. \square

5.2. The partition of the derived category by Levis.

Definition 5.2. We define $\mathbf{D}(\mathfrak{g})_{\nless(L)}$ (respectively $\mathbf{D}(\mathfrak{g})_{\leq(L)}$) as the full subcategory of $\mathbf{D}(\mathfrak{g})$ consisting of objects such that both \mathfrak{M} and $\mathbb{T}_{\mathfrak{g}}\mathfrak{M}$ are contained in the closed substack $\mathfrak{g}_{\nless(L)}$ (respectively $\mathfrak{g}_{\leq(L)}$).

According to Proposition 5.1, the subcategory $\mathbf{D}(\mathfrak{g})_{\nless(L)}$ can be identified with the kernel of $\mathbf{Res}_{P,L}^G$ for any choice of parabolic P containing L . Similarly, $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ consists of objects with are killed by parabolic restriction to proper Levi subgroups of L . In the special case $L = G$, then

$\mathbf{D}(\underline{\mathfrak{g}})_{\nless(G)}$ consists of objects which are killed by parabolic restriction for any proper parabolic of G . We will refer to such objects as *cuspidal*, and we write $\mathbf{D}(\underline{\mathfrak{g}})_{\text{cusp}} := \mathbf{D}(\underline{\mathfrak{g}})_{\nless(G)}$.

Definition 5.3. We define $\mathbf{D}(\underline{\mathfrak{g}})_{\leq(L)}$ as the left orthogonal to $\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)}$. Similarly, $\mathbf{D}(\underline{\mathfrak{g}})_{(L)}$ is defined to be the left orthogonal to $\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)}$ in $\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)}$.

Recall that we have a continuous adjunction:

$$(11) \quad \mathbf{Ind}_{P,L}^G : \mathbf{D}(\underline{\mathfrak{l}}) \rightleftarrows \mathbf{D}(\underline{\mathfrak{g}}) : \mathbf{Res}_{P,L}^G$$

Note that an object $\mathfrak{M} \in \mathbf{D}(\underline{\mathfrak{g}})$ is contained in $\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)}$ if and only if $\mathbf{Res}_{P,L}^G(\mathfrak{M})$ is cuspidal. On the other hand, if $\mathfrak{N} \in \mathbf{D}(\underline{\mathfrak{l}})_{\text{cusp}}$, then it follows from the Mackey formula, Proposition 3.5 that $\mathbf{Ind}_{P,L}^G(\mathfrak{N})$ is contained in $\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)}$. Thus we have another continuous adjunction.

$$(12) \quad \mathbf{Ind}_{P,L}^G|_{\text{cusp}} : \mathbf{D}(\underline{\mathfrak{l}})_{\text{cusp}} \rightleftarrows \mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)} : \mathbf{Res}_{P,L}^G|_{\nless(L)}.$$

We may apply the results of Subsection 2.2 to either of the adjunctions 11 or 12 above to obtain the following:

Proposition 5.4. For each Levi subgroup L of G , we have equivalences $\mathbf{D}(\underline{\mathfrak{g}})_{\leq(L)} \simeq \mathbf{D}(\underline{\mathfrak{l}})^{\text{St}}$ and $\mathbf{D}(\underline{\mathfrak{g}})_{(L)} \simeq \mathbf{D}(\underline{\mathfrak{l}})_{\text{cusp}}^{\text{St}}$, and diagrams

$$\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)} \begin{array}{c} \xleftarrow{i_{\leq(L)}^*} \\ \xrightarrow{i_{\leq(L)*}} \end{array} \mathbf{D}(\underline{\mathfrak{g}}) \begin{array}{c} \xleftarrow{j_{\leq(L)}^!} \\ \xrightarrow{j_{\leq(L)}!} \end{array} \mathbf{D}(\underline{\mathfrak{l}})^{\text{St}},$$

and

$$\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)} \begin{array}{c} \xleftarrow{i_{(L)*}} \\ \xrightarrow{i_{(L)*}} \end{array} \mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)} \begin{array}{c} \xleftarrow{j_{(L)}^!} \\ \xrightarrow{j_{(L)}!} \end{array} \mathbf{D}(\underline{\mathfrak{l}})_{\text{cusp}}^{\text{St}},$$

where:

- (1) the canonical functor $j_{\leq(L)}^!$ admits a left adjoint $j_{\leq(L)}!$;
- (2) the inclusion functor $i_{\leq(L)*}$ admits a left adjoint $i_{\leq(L)}^*$;
- (3) The counit $i_{\leq(L)}^* i_{\leq(L)*} \rightarrow 1_{\mathbf{D}(\underline{\mathfrak{g}})_{\nless(L)}}$, and the unit $1_{\mathbf{D}(\underline{\mathfrak{g}})_{\leq(L)}} \rightarrow j_{\leq(L)}^! j_{\leq(L)}!$ are isomorphisms;
- (4) There is a distinguished triangle of functors

$$j_{\leq(L)}! j_{\leq(L)}^! \rightarrow 1_{\mathbf{D}(\underline{\mathfrak{g}})} \rightarrow i_{\leq(L)*} i_{\leq(L)}^* \xrightarrow{+1};$$

- (5) the (fully faithful) essential image of the functor $j_{\leq(L)}$ is precisely the subcategory $\mathbf{D}(\underline{\mathfrak{g}})_{\leq(L)}$ (which also is equal to the cocompletion of the essential image of $\mathbf{Ind}_{P,L}^G$).

(Analogous statements to (1)-(5) hold for the functors $i_{(L)*}$, $j_{(L)}^!$, etc.)

Remark 5.5. We have the following formula for $j_{\leq(L)}! j_{\leq(L)}^!$:

$$j_{\leq(L)}! j_{\leq(L)}^!(\mathfrak{M}) = \text{colim} \left(\mathbf{Ind}_{P,L}^G \mathbf{Res}_{P,L}^G(\mathfrak{M}) \rightleftarrows \mathbf{Ind}_{P,L}^G \mathbf{Res}_{P,L}^G \mathbf{Ind}_{P,L}^G \mathbf{Res}_{P,L}^G(\mathfrak{M}) \rightleftarrows \dots \right)$$

Note that the functors $\mathbf{Res}_{P,L}^G$ and $\mathbf{Ind}_{P,L}^G$ may depend on the choice of parabolic P in general, even though the functors $j_{\leq(L)}^!$ and $j_{\leq(L)}!$ are independent of this choice.

5.3. Orthogonality.

Proposition 5.6. *Let P, Q, M, L be as in Subsection 3.1, and suppose that M is not conjugate to L . Then given $\mathfrak{N} \in \mathbf{D}_{coh}^b(\mathbb{I})_{cusp}$, and $\mathfrak{M} \in \mathbf{D}_{coh}^b(\underline{\mathfrak{m}})_{cusp}$ we have that $\mathbf{Ind}_{P,L}^G(\mathfrak{N})$ and $\mathbf{Ind}_{Q,L}^G(\mathfrak{M})$ are orthogonal, i.e.*

$$R\mathrm{Hom}(\mathbf{Ind}_{P,L}^G(\mathfrak{N}), \mathbf{Ind}_{Q,L}^G(\mathfrak{M})) \simeq R\mathrm{Hom}(\mathbf{Ind}_{Q,L}^G(\mathfrak{M}), \mathbf{Ind}_{P,L}^G(\mathfrak{N})) \simeq 0.$$

Proof. First let first assume L is not conjugate to a subgroup of M . Let $\mathfrak{N} \in \mathbf{D}_{coh}^b(\mathbb{I})_{cusp}$ and $\mathfrak{M} \in \mathbf{D}_{coh}^b(\underline{\mathfrak{m}})_{cusp}$. As M is not conjugate to a subgroup of L , for any $g \in G$, $M \cap {}^g L$ is a proper subgroup of ${}^g L$. Thus

$${}_{Q,M}G\mathrm{St}_{P,L}^w \mathfrak{N} \simeq \mathbf{Ind}_{M \cap {}^w P, M \cap {}^w L}^M \mathbf{Res}_{Q \cap {}^w L, M \cap {}^w L}^{{}^w L} {}^w \mathfrak{N} \simeq 0$$

as ${}^w \mathfrak{N}$ is cuspidal for each w . By the Proposition 3.5 (the Mackey decomposition), ${}_{Q,M}G\mathrm{St}_{P,L} \mathfrak{N} \simeq 0$. Thus

$$(13) \quad R\mathrm{Hom}(\mathbf{Ind}_{Q,M}^G \mathfrak{M}, \mathbf{Ind}_P^G \mathfrak{N}) = R\mathrm{Hom}(\mathfrak{M}, {}_{Q,M}G\mathrm{St}_{P,L} \mathfrak{N}) \simeq 0.$$

Note that the Verdier duality functor $\mathbb{D}_{\mathbb{I}}$ preserves the category of cuspidal objects in $\mathbf{D}(\mathbb{I})$ and the functor $\mathbf{Ind}_{P,L}^G$ intertwines the Verdier duality functors. Applying 13 with \mathfrak{N} and \mathfrak{M} replaced by $\mathbb{D}_{\mathbb{I}} \mathfrak{N}$ and $\mathbb{D}_{\underline{\mathfrak{m}}} \mathfrak{M}$, we obtain:

$$\begin{aligned} R\mathrm{Hom}(\mathbf{Ind}_{P,L}^G \mathfrak{N}, \mathbf{Ind}_{Q,M}^G \mathfrak{M}) &= R\mathrm{Hom}(\mathbb{D}_{\underline{\mathfrak{g}}} \mathbf{Ind}_{Q,M}^G \mathfrak{M}, \mathbb{D}_{\underline{\mathfrak{g}}} \mathbf{Ind}_{P,L}^G \mathfrak{N}) \\ &= R\mathrm{Hom}(\mathbf{Ind}_{Q,M}^G \mathbb{D}_{\underline{\mathfrak{m}}} \mathfrak{M}, \mathbf{Ind}_{P,L}^G \mathbb{D}_{\mathbb{I}} \mathfrak{N}) = 0, \end{aligned}$$

Thus $\mathbf{Ind}_{P,L}^G(\mathfrak{N})$ is orthogonal to $\mathbf{Ind}_{Q,M}^G(\mathfrak{M})$ whenever M is not conjugate to a subgroup of L . By switching the roles of M and L , we obtain that $\mathbf{Ind}_{P,L}^G(\mathfrak{N})$ and $\mathbf{Ind}_{Q,M}^G(\mathfrak{M})$ are also orthogonal whenever L is not conjugate to a subgroup of M . Thus $\mathbf{Ind}_{P,L}^G(\mathfrak{N})$ and $\mathbf{Ind}_{Q,M}^G(\mathfrak{M})$ are orthogonal whenever M is not conjugate to L , as required. \square

5.4. Remark on the proof of Theorem 0.3. In order to prove Theorem 0.3, we need to show that $\mathbf{D}(\underline{\mathfrak{g}})$ decomposes as an orthogonal direct sum of the subcategories $\mathbf{D}(\underline{\mathfrak{g}})_{(L)}$, as L ranges over conjugacy classes of Levi subgroups of G .

Given any object $\mathfrak{M} \in \mathbf{D}(\underline{\mathfrak{g}})$, we can write it as an iterated extension of objects of $\mathbf{D}(\underline{\mathfrak{g}})_{(L)}$ as follows:

- (1) First choose a Levi subgroup L which is minimal such that $\mathbf{Res}_{P,L}^G(\mathfrak{M}) \neq 0$ (thus $\mathfrak{M} \in \mathbf{D}(\underline{\mathfrak{g}})_{\neq(L)}$).
- (2) We have a distinguished triangle:

$$\mathbf{i}_{(L)*} \mathbf{i}_{(L)}^* \mathfrak{M} \rightarrow \mathfrak{M} \rightarrow \mathbf{j}_{(L)!} \mathbf{j}_{(L)}^! \mathfrak{M} \xrightarrow{+1},$$

- (3) Replace \mathfrak{M} by $\mathbf{i}_{(L)*} \mathbf{i}_{(L)}^* (\mathfrak{M})$ and repeat steps (1) and (2) (the algorithm halts after finitely many steps when $L = G$).

Let $\langle \mathbf{Ind}_{P,L}^G(\mathbf{D}_{coh}^b(\mathbb{I})_{cusp}) \rangle$ denote the Karoubian envelope of the objects $\mathbf{Ind}_{P,L}^G(\mathfrak{M})$ where $\mathfrak{M} \in \mathbf{D}_{coh}^b(\mathbb{I})_{cusp}$. It follows immediately from Proposition 5.6 that $\langle \mathbf{Ind}_{P,L}^G(\mathbf{D}_{coh}^b(\mathbb{I})_{cusp}) \rangle$ is orthogonal to $\langle \mathbf{Ind}_{Q,M}^G(\mathbf{D}_{coh}^b(\underline{\mathfrak{m}})_{cusp}) \rangle$, whenever L is not conjugate to M . In order to deduce the orthogonal decomposition of $\mathbf{D}(\underline{\mathfrak{g}})$ into $\mathbf{D}(\underline{\mathfrak{g}})_{(L)}$, it remains to show the following:

Lemma 5.7. *For every Levi subgroup L of G , we have:*

- (1) the functor $j_{(L)!}j_{(L)}^!$ takes bounded coherent complexes to bounded coherent complexes;
- (2) The subcategory $\mathbf{D}_{coh}^b(\mathfrak{g})_{(L)}$ of $\mathbf{D}(\mathfrak{g})_{(L)}$ consisting of bounded coherent complexes is equal to the Karoubian envelope $\langle \mathbf{Ind}_{P,L}^G(\mathbf{D}_{coh}^b(\mathfrak{l})_{cusp}) \rangle$.

Indeed, to deduce Theorem 0.3 from Lemma 5.7, observe that given $\mathfrak{M} \in \mathbf{D}_{coh}^b(\mathfrak{g})$, we can use the algorithm above to write \mathfrak{M} as an iterated extension of objects in $\mathbf{D}(\mathfrak{g})_{(L)}$; by Lemma 5.7 part 1, these objects are also bounded coherent, so by Lemma 5.7 part 2 and Proposition 5.6 these extensions must all split. Thus $\mathbf{D}_{coh}^b(\mathfrak{g})$ splits as a direct sum of $\mathbf{D}_{coh}^b(\mathfrak{g})_{(L)}$. As these subcategories contain a set of compact generators, the decomposition of $\mathbf{D}(\mathfrak{g})$ follows.

In fact, Lemma 5.7 is also *necessary* for an orthogonal decomposition, as the following example shows. Consider the category $\mathbf{D}(\mathcal{N}_G; \mathbb{F})$ of G -equivariant complexes of sheaves of \mathbb{F} vector spaces on the nilpotent cone \mathcal{N}_G , where \mathbb{F} is a finite field. This category has induction and restriction functors, which enjoy all the same properties as we have proved in the D -module (i.e. they are exact, preserve constructible complexes, commute with Fourier transform, etc.) with the exception of Lemma 5.7. In fact, one can already see the failure of Lemma 5.7 (and indeed Theorem 0.3) in the case $G = SL_2$ and $\mathbb{F} = \mathbb{F}_2$. The author would like to thank C. Mautner for first drawing this example to his attention.

6. THE ABELIAN DECOMPOSITION AND SPRINGER THEORY

6.1. Partition of the Abelian Category by Levis. For each Levi subgroup L , we have the subcategories $\mathbf{M}(\mathfrak{g})_{\leq(L)}$ (respectively, $\mathbf{M}(\mathfrak{g})_{\neq(L)}$), defined as the intersection of $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ (respectively $\mathbf{D}(\mathfrak{g})_{\neq(L)}$) with $\mathbf{M}(\mathfrak{g})$. By Proposition 5.1, $\mathbf{M}(\mathfrak{g})_{\leq(L)}$ can be identified with the kernel of $\mathbf{res}_{P,L}^G$ and $\mathbf{M}(\mathfrak{g})_{\neq(L)}$ is the subcategory consisting of objects \mathfrak{M} such that $\mathbf{res}_{P,L}^G(\mathfrak{M})$ is cuspidal.

Proposition 6.1. *For each Levi subgroup L of G , we have colocalization diagrams:*

$$\mathbf{M}(\mathfrak{g})_{\leq(L)} \begin{matrix} \xleftarrow{i_{\leq(L)}^*} \\ \xrightarrow{i_{\leq(L)}^*} \end{matrix} \mathbf{M}(\mathfrak{g}) \begin{matrix} \xleftarrow{j_{\leq(L)}^!} \\ \xrightarrow{j_{\leq(L)}^!} \end{matrix} \mathbf{M}(\mathfrak{g})_{\leq(L)},$$

and

$$\mathbf{M}(\mathfrak{g})_{\leq(L)} \begin{matrix} \xleftarrow{i_{(L)}^*} \\ \xrightarrow{i_{(L)}^*} \end{matrix} \mathbf{M}(\mathfrak{g})_{\neq(L)} \begin{matrix} \xleftarrow{j_{(L)}^!} \\ \xrightarrow{j_{(L)}^!} \end{matrix} \mathbf{M}(\mathfrak{g})_{(L)},$$

where:

- (1) the functor $j_{\leq(L)!}$ is left adjoint to $j_{\leq(L)}^!$;
- (2) the functor $i_{\leq(L)}^*$ is left adjoint to $i_{\leq(L)}^*$;
- (3) The counit $i_{\leq(L)}^* i_{\leq(L)}^* \rightarrow 1_{\mathbf{D}(\mathfrak{g})_{\leq(L)}}$, and the unit $1_{\mathbf{D}(\mathfrak{g})_{\leq(L)}} \rightarrow j_{\leq(L)}^! j_{\leq(L)!}$ are isomorphisms;
- (4) There is a short exact sequence of functors

$$0 \rightarrow j_{\leq(L)!} j_{\leq(L)}^! \rightarrow 1_{\mathbf{D}(\mathfrak{g})} \rightarrow i_{\leq(L)}^* i_{\leq(L)}^* \rightarrow 0;$$

- (5) the subcategory $\mathbf{D}(\mathfrak{g})_{\leq(L)}$ is the cocompletion of the essential image of $\mathbf{Ind}_{P,L}^G$.

(Analogous statements to (1)-(5) hold for the functors $i_{(L)}^*$, $j_{(L)}^!$, etc.)

Remark 6.2. Explicitly, the object $j_{\leq(L)!} j_{\leq(L)}^! \mathfrak{N}$ can be computed as the coequalizer of the diagram:

$$\mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G \mathfrak{N} \rightrightarrows \mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G \mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G \mathfrak{N}$$

Unlike in the derived category it is clear that if \mathfrak{M} is coherent, then $j_{\leq(L)}! j_{\leq(L)}^! \mathfrak{M}$ is also coherent. At this stage it has not been shown that the functor $j_{\leq(L)}! j_{\leq(L)}^!$ is exact (we will see that it is in fact exact in Corollary 6.4).

6.2. The Steinberg and the Weyl Monad. Let us fix P, L and consider the monad $\mathbf{st} = {}_{L,P}^G \mathbf{st}^{P,L}$ acting on $\mathbf{M}(\mathfrak{l})$, and its restriction $\mathbf{st}|_{\text{cusp}}$ acting on $\mathbf{M}(\mathfrak{l})_{\text{cusp}}$. Recall that the finite group $W(G, L)$ acts on the stack \mathfrak{l} , and thus on the categories $\mathbf{M}(\mathfrak{l})$ and $\mathbf{M}(\mathfrak{l})_{\text{cusp}}$. This action gives rise to monads $W(G, L)_*$ and $W(G, L)_*|_{\text{cusp}}$ acting on $\mathbf{M}(\mathfrak{l})$ and $\mathbf{M}(\mathfrak{l})_{\text{cusp}}$.

Theorem 6.3. *There is an equivalence of monads $\mathbf{st}|_{\text{cusp}} \simeq W(G, L)_*|_{\text{cusp}}$ acting on $\mathbf{M}(\mathfrak{l})_{\text{cusp}}$.*

The proof of Theorem 6.3 will be given in Subsection 6.4.

Corollary 6.4.

- (1) *Given $\mathfrak{M} \in \mathbf{M}(\mathfrak{g})_{\star(L)}$, we have that $\mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G(\mathfrak{M})$ carries an action of $W(G, L)$ such that:*

$$j_{(L)}! j_{(L)}^! (\mathfrak{M}) = \left(\mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G(\mathfrak{M}) \right)_W$$

- (2) *The functor*

$$j_{(L)}! j_{(L)}^! : \mathbf{M}(\mathfrak{g})_{\star(L)} \rightarrow \mathbf{M}(\mathfrak{g})_{(L)}$$

is exact.

- (3) *The subcategory $\mathbf{M}(\mathfrak{g})_{(L)}$ consists precisely of direct summands of objects of the form $\mathbf{ind}_{P,L}^G(\mathfrak{M})$ where $\mathfrak{M} \in \mathbf{M}(\mathfrak{l})$ is cuspidal.*

Proof. The first claim follows immediately from the results of Subsection 2.3, and the other claims follow from the fact that the coinvariants $\left(\mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G(\mathfrak{M}) \right)_W$ are a direct summand of $\mathbf{ind}_{P,L}^G \mathbf{res}_{P,L}^G(\mathfrak{M})$, and the functor of taking coinvariants is exact. \square

6.3. The Generalized Springer Correspondence. According to Theorem 6.3 and the Barr-Beck Theorem we have that

$$\mathbf{M}(\mathfrak{l})_{(L)} \simeq \mathbf{M}(\mathfrak{l})_{\text{cusp}}^{W(G,L)}.$$

Recall that

$$\mathbf{M}(\mathfrak{l})_{\text{cusp}} = \bigoplus_{(\mathcal{O}_L, \mathcal{E}_L)} \mathbf{M}(\mathfrak{z}(\mathfrak{l})) \boxtimes IC(\mathcal{O}_L, \mathcal{E}_L),$$

as $(\mathcal{O}_L, \mathcal{E}_L)$ ranges over the set of cuspidal pairs for L .

Proposition 6.5. *Given a simple object $\mathfrak{E} \in \mathbf{M}(\mathfrak{l})_{\text{cusp}}$, the following conditions are equivalent:*

- (1) *the object \mathfrak{E} carries a $W(G, L)$ -equivariant structure;*
 (2) *there is an object $\mathfrak{F} \in \mathbf{M}(\mathcal{N}_G)$ such that $\mathbf{res}_{P,L}^G(\mathfrak{F}) \simeq \mathfrak{E}$.*

Moreover, if either of the above conditions hold, then

$$\text{End}(\mathbf{ind}_{P,L}^G(\mathfrak{E})) \simeq \mathbb{C}[W]$$

Proof. This is a special case of Subsection 2.3. \square

Theorem 6.6 (Lusztig). *Every object \mathfrak{E} of $\mathbf{M}(\mathcal{N}_L)_{\text{cusp}}$ satisfies the conditions of Proposition 6.5.*

Remark 6.7. Recall that if $\mathfrak{E} = IC(\mathcal{Q}, \mathcal{E}) \in \mathbf{M}(\underline{\mathcal{N}}_L)$ is cuspidal, then the nilpotent orbit \mathcal{O} is distinguished. There are two canonical nilpotent orbits of G associated to \mathcal{O} : namely, \mathcal{O}_G^{ind} which is characterized by the condition that $\mathcal{O}_G \cap \mathfrak{u}$ is open and dense, and \mathcal{O}_G^{cont} which is characterized by the condition that $\mathcal{O}_G^{cont} \cap \mathcal{O}_L \neq \emptyset$. There are two canonical summands $\mathfrak{F}^{ind} = IC(\mathcal{O}_G^{ind}, \mathcal{E}^{ind})$ and $\mathfrak{F}^{cont} = IC(\mathcal{O}_G^{cont}, \mathcal{E})$ such that $\mathbf{res}_{P,L}^G(\mathfrak{F}^{ind}) \simeq \mathfrak{E} \simeq \mathbf{res}_{P,L}^G(\mathfrak{F}^{cont})$. Thus the object \mathfrak{E} carries two canonical $W(G, L)$ equivariant structures.

We define $\mathbf{M}(\underline{\mathfrak{g}})_{(L, \mathcal{O}_L, \mathcal{E}_L)}$ to be the subcategory of $\mathbf{M}(\underline{\mathfrak{g}})$ generated by objects \mathfrak{M} such that

$$\mathbf{res}_{P,L}^G(\mathfrak{M}) \in \mathbf{M}(\mathfrak{z}(\mathfrak{l})) \boxtimes IC(\mathcal{O}_L, \mathcal{E}_L).$$

Corollary 6.8.

(1)

$$\mathbf{M}(\underline{\mathfrak{g}})_{(L)} \simeq \bigoplus_{(\mathcal{O}_L, \mathcal{E}_L)} \mathbf{M}(\underline{\mathfrak{g}})_{(L, \mathcal{O}_L, \mathcal{E}_L)}.$$

(2)

$$\mathbf{M}(\underline{\mathfrak{g}})_{(L, \mathcal{O}_L, \mathcal{E}_L)} \simeq \mathbf{M}(\mathfrak{z}(\mathfrak{l}))^{W(G, L)}.$$

Proof. By construction, every object of $\mathbf{M}(\underline{\mathcal{N}}_L)_{(L)}$ is a direct sum of objects of $\mathbf{M}(\underline{\mathcal{N}}_L)_{(L, \mathcal{O}_L, \mathcal{E}_L)}$; moreover, if $(\mathcal{O}_L, \mathcal{E}_L)$ is not isomorphic to $(\mathcal{O}'_L, \mathcal{E}'_L)$ then the subcategories $\mathbf{M}(\underline{\mathcal{N}}_L)_{(L, \mathcal{O}_L, \mathcal{E}_L)}$ and $\mathbf{M}(\underline{\mathcal{N}}_L)_{(L, \mathcal{O}'_L, \mathcal{E}'_L)}$ are orthogonal. Indeed, we have:

$$\mathrm{Hom}(\mathbf{ind}_{P,L}^G \mathfrak{E}, \mathbf{ind}_{P,L}^G \mathfrak{E}') \simeq \mathrm{Hom}(\mathfrak{E}, \bigoplus_{w \in W} w_* \mathfrak{E}') \simeq 0,$$

and thus the space of morphisms between summands of $\mathbf{ind}_{P,L}^G \mathfrak{E}$ and of $\mathbf{ind}_{P,L}^G \mathfrak{E}'$ must also be trivial. The second claim follows from the Barr-Beck theorem. \square

6.4. Proof of Theorem 6.3. Consider the category $\mathrm{Fun}^c(\mathbf{M}(\underline{\mathfrak{l}})_{cusp}, \mathbf{M}(\underline{\mathfrak{l}})_{cusp})$ of right exact endofunctors of the abelian category $\mathbf{M}(\underline{\mathfrak{l}})_{cusp}$. To show that the Mackey filtration can be split, it is enough to show the following:

Lemma 6.9. *We have*

$$\mathrm{Ext}^i(v_*, w_*) = 0,$$

where v_*, w_* are considered objects of $\mathrm{Fun}^c(\mathbf{M}(\underline{\mathfrak{l}})_{cusp}, \mathbf{M}(\underline{\mathfrak{l}})_{cusp})$.

To prove Lemma 6.9, we will compute the Ext groups in terms of the corresponding integral transforms to the functors. Recall that if $\mathfrak{N} \in \mathbf{M}(\underline{\mathfrak{l}})$ is cuspidal, it is supported on $\mathfrak{l}^\heartsuit \subseteq \mathfrak{l}$, where

$$\mathfrak{l}^\heartsuit \simeq \underline{\mathcal{N}}_L \times \mathfrak{z}(\mathfrak{l}).$$

Thus we have an equivalence of categories

$$\mathbf{M}(\mathfrak{l}^\heartsuit) \simeq \mathbf{M}(\underline{\mathcal{N}}_L) \otimes \mathbf{M}(\mathfrak{z}(\mathfrak{l})).$$

Note also that the category $\mathbf{M}(\underline{\mathcal{N}}_L)$ is semisimple with finitely many simple objects; thus the category $\mathbf{M}(\underline{\mathfrak{l}})_{cusp}$ is a direct sum

$$\bigoplus_{(\mathcal{O}, \mathcal{E})} \mathbf{M}(\mathfrak{z}(\mathfrak{l})) \boxtimes IC(\mathcal{O}, \mathcal{E}).$$

where the sum is indexed by isomorphism classes of nilpotent orbits \mathcal{O} in \mathcal{N}_L together with a cuspidal local system \mathcal{E} on \mathcal{O} . (Abstractly, the category $\mathbf{M}(\underline{\mathfrak{l}})_{cusp}$ is equivalent to a direct sum of copies of $\mathbf{M}(\mathfrak{z}(\mathfrak{l})) \simeq \mathfrak{D}_{\mathfrak{z}(\mathfrak{l})} - \mathrm{mod}$.)

For simplicity, let us assume that there is only one cuspidal pair $(\mathcal{O}, \mathcal{E})$ (the argument looks a little more messy in general). Thus we are reduced to showing that there are no non-trivial extensions between the functors

$$w_*, v_* : \mathbf{M}(\mathfrak{z}(\mathfrak{l})) \rightarrow \mathbf{M}(\mathfrak{z}(\mathfrak{l}))$$

if $w \neq v$ are in $W(G, L)$. Note that right exact endofunctors of $\mathbf{M}(\mathfrak{z}(\mathfrak{l}))$ can be identified with $\mathfrak{D}_{\mathfrak{z}(\mathfrak{l})}$ -bimodules, which in turn, we identify with objects of $\mathbf{M}(\mathfrak{z}(\mathfrak{l}) \times \mathfrak{z}(\mathfrak{l}))$ (we call the object in $\mathbf{M}(\mathfrak{z}(\mathfrak{l}) \times \mathfrak{z}(\mathfrak{l}))$ the integral kernel corresponding to the given functor). For each $w \in W(G, L)$, let $\mathcal{O}_w := \Gamma_{w*} \mathcal{O}_{\mathfrak{z}}$ where $\Gamma_w : \mathfrak{z} \rightarrow \mathfrak{z} \times \mathfrak{z}$ is the graph of the $W(G, L)$ -action; this is the integral kernel corresponding to the functor w_* .

Lemma 6.9 thus follows from Lemma 6.10 below. For the following lemma, to simplify notation we write $\mathfrak{z} := \mathfrak{z}(\mathfrak{l})$, $W = W(G, L)$.

Lemma 6.10. *Let $v, w \in W$, $v \neq w$. Then*

$$\mathrm{Ext}_{\mathbf{M}(\mathfrak{z} \times \mathfrak{z})}^i(\mathcal{O}_v, \mathcal{O}_w) = 0,$$

for $i = 0, 1$.

Proof. Let $\mathfrak{z}^{v,w} = \{x \in \mathfrak{z} \mid v(x) = w(x)\}$. We have the cartesian diagram:

$$\begin{array}{ccc} \mathfrak{z}^{v,w} & \xrightarrow{\gamma_w} & \mathfrak{z} \\ \gamma_v \downarrow & & \downarrow \Gamma_v \\ \mathfrak{z} & \xrightarrow{\Gamma_w} & \mathfrak{z} \times \mathfrak{z} \end{array}$$

where $\gamma_w = \gamma_v$ is the inclusion of $\mathfrak{z}^{v,w}$ into \mathfrak{z} . By our assumption $w \neq v$, $\mathfrak{z}^{v,w} \neq \mathfrak{z}$. Let d denote the codimension of $\mathfrak{z}^{v,w}$ in \mathfrak{z} . We compute:

$$\begin{aligned} R\mathrm{Hom}_{\mathbf{D}(\mathfrak{z} \times \mathfrak{z})}(\mathcal{O}_v, \mathcal{O}_w) &= R\mathrm{Hom}_{\mathbf{D}(\mathfrak{z} \times \mathfrak{z})}(\Gamma_{v*} \mathcal{O}_{\mathfrak{z}}, \Gamma_{w*} \mathcal{O}_{\mathfrak{z}}) \\ &= R\mathrm{Hom}_{\mathbf{D}(\mathfrak{z})}(\mathcal{O}_{\mathfrak{z}}, \Gamma_v^! \Gamma_w^* \mathcal{O}_{\mathfrak{z}}) \\ &= R\mathrm{Hom}_{\mathbf{D}(\mathfrak{z})}(\mathcal{O}_{\mathfrak{z}}, \gamma_{w*} \gamma_v^! \mathcal{O}_{\mathfrak{z}}) \\ &= R\mathrm{Hom}_{\mathbf{D}(\mathfrak{z}^{v,w})}(\gamma_w^* \mathcal{O}_{\mathfrak{z}}, \gamma_v^! \mathcal{O}_{\mathfrak{z}}) \\ &= R\mathrm{Hom}_{\mathbf{D}(\mathfrak{z}^{v,w})}(\mathcal{O}_{\mathfrak{z}^{v,w}}[d], \mathcal{O}_{\mathfrak{z}^{v,w}}[-d]) = H^*(\mathfrak{z}^{v,w}; \mathbb{C})[-2d]. \end{aligned}$$

As $d > 0$, $\mathrm{Ext}^1(\mathcal{O}_v, \mathcal{O}_{\mathfrak{z}(w)}) = H^1(H^*(\mathfrak{z}^{v,w}; \mathbb{C})[-2d]) = 0$ as required. \square

It follows from Lemma 6.9 that the filtration on $\mathbf{st}|_{\mathrm{cusp}}$ is split, i.e. there exists a natural isomorphism of functors

$$(14) \quad \mathbf{st}|_{\mathrm{cusp}} \simeq W(G, L)_*|_{\mathrm{cusp}}.$$

To complete the proof of Theorem 6.3, we must show that we can choose the natural isomorphism 14 to be an isomorphism of monads. Observe that by Corollary 4.8, we have the required isomorphism of monads after restricting to the regular locus:

$$(15) \quad \phi^{\mathrm{reg}} : W(G, L)_*|_{\mathbf{M}(\mathbb{L}^{\mathrm{reg}})_{\mathrm{cusp}}} \simeq \mathbf{st}|_{\mathbf{M}(\mathbb{L}^{\mathrm{reg}})_{\mathrm{cusp}}}.$$

Note also that the restriction functor

$$\mathrm{Hom}_{\mathbf{M}(\mathfrak{z}(\mathfrak{l}) \times \mathfrak{z}(\mathfrak{l}))}(\mathcal{O}_v, \mathcal{O}_w) \rightarrow \mathrm{Hom}_{\mathbf{M}(\mathfrak{z}(\mathfrak{l})^{\mathrm{reg}} \times \mathfrak{z}(\mathfrak{l})^{\mathrm{reg}})}(\mathcal{O}_v|_{\mathrm{reg}}, \mathcal{O}_w|_{\mathrm{reg}}),$$

is an isomorphism. Thus the isomorphism from 15 extends uniquely to:

$$\phi : W(L)^*|_{\text{cusp}} \simeq \mathbf{st}|_{\text{cusp}}.$$

The monad structure on the functor $W(G, L)_*|_{\text{cusp}}$ is given by a collection of isomorphisms

$$\tau_{v,w} : w_*v_* \xrightarrow{\sim} (wv)_*,$$

for each pair $(v, w) \in W(G, L)^2$. Similarly, the monad structure on $\mathbf{st}|_{\text{cusp}}$ induces a (a priori different) monad structure on $W(G, L)_*$ via the natural isomorphism ϕ ; we denote the corresponding structure morphisms by $\tau'_{v,w}$. Corollary 4.8 implies that $\tau = \tau'$ after restricting to the regular locus. On the other hand, the restriction map

$$\text{Hom}(w_*v_*, (wv)_*) \xrightarrow{\text{Hom}} (w_*v_*|_{\text{reg}}, (wv)_*|_{\text{reg}})$$

is an isomorphism. Thus $\tau = \tau'$ and ϕ is an isomorphism of monads as required. This completes the proof of Theorem 6.3.

6.5. Proof of Lemma 5.7. Recall that the first claim of Lemma 5.7 is that the functor

$$j_{(L)}^! j_{(L)}! : \mathbf{D}(\underline{\mathfrak{g}})_{\dagger(L)} \rightarrow \mathbf{D}(\underline{\mathfrak{g}})_{(L)}$$

takes bounded coherent complexes to bounded coherent complexes. By Corollary 6.4, the functor $j_{(L)}^! j_{(L)}!$ takes coherent objects in the heart of the t -structure to coherent objects in the heart of the t -structure. As every object of $\mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{g}})_{(L)}$ is generated by shifts and cones of objects in the heart of the t -structure, the result follows.

The second claim is that $\mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{g}})_{(L)}$ is generated by summands, shifts, and cones of objects of the form $\mathbf{Ind}_{P,L}^G(\mathfrak{M})$, where $\mathfrak{M} \in \mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{l}})_{\text{cusp}}$. Again, for objects in the heart of the t -structure, this follows from Corollary 6.4, and thus it holds for all of $\mathbf{D}_{\text{coh}}^b(\underline{\mathfrak{g}})_{(L)}$.

6.6. Proof of Theorem 0.3. As explained in Subsection 5.4, we now can deduce that $\mathbf{D}(\underline{\mathfrak{g}})$ decomposes as an orthogonal sum of the subcategories $\mathbf{D}(\underline{\mathfrak{g}})_{(L)}$, and

$$\mathbf{D}(\underline{\mathfrak{g}})_{(L)} \simeq \mathbf{D}(\underline{\mathfrak{l}})^{\text{St}}.$$

To deduce the further decomposition stated in Theorem 0.3, we need the following important result of Lusztig. Recall that given a quotient stack X , a pair (U, \mathcal{L}) of a smooth substack $j : U \hookrightarrow X$ and a local system \mathcal{L} on U is called *clean* if the natural map

$$j_! \mathcal{L} \rightarrow j_* \mathcal{L}$$

is an isomorphism. (In that case, we have $j_! \mathcal{L} \simeq IC(U; \mathcal{L}) \simeq j_* \mathcal{L}$).

Theorem 6.11. *Irreducible objects of $\mathbf{M}(\underline{\mathcal{N}}_L)_{\text{cusp}}$ are clean.*

It follows that if $\mathfrak{E}_1 = IC(\mathcal{O}_1, \mathcal{E}_1)$ and $\mathfrak{E}_2 = IC(\mathcal{O}_2, \mathcal{E}_2)$ are two irreducible cuspidal objects, then

$$R\text{Hom}(\mathfrak{E}_1, \mathfrak{E}_2) \simeq \begin{cases} \mathbb{C} & \text{if } (\mathcal{O}_1, \mathcal{E}_1) \simeq (\mathcal{O}_2, \mathcal{E}_2) \\ 0 & \text{otherwise..} \end{cases}$$

Indeed, if $\mathcal{O}_1 \neq \mathcal{O}_2$, then

$$R\text{Hom}(\mathfrak{E}_1, \mathfrak{E}_2) \simeq R\text{Hom}(j_{1!} \mathcal{E}_1, j_{2*} \mathcal{E}_2) \simeq R\text{Hom}(j_2^* j_{1!} \mathcal{E}_1, \mathcal{E}_2) \simeq 0.$$

Suppose $\mathcal{O}_1 = \mathcal{O}_2 =: \mathcal{O}$. Recall that \mathcal{O} is distinguished: this means that for each $x \in \mathcal{O}$, $C_L^0(x)$ is unipotent. Thus the differential graded algebra of chains $C_*(C_L(x); \mathbb{C})$ is quasi isomorphic to

the group algebra of $A_L(x) = C_L(x)/C_L^\circ(x)$. It follows that the category $\mathbf{D}(\mathcal{Q}) \simeq \mathbf{D}(pt/C_G(x))$ is equivalent to the category of differential graded modules for the algebra $\mathbb{C}[A_L(x)]$. In particular,

$$R\mathrm{Hom}(\mathfrak{E}_1, \mathfrak{E}_2) \simeq R\mathrm{Hom}(j_*\mathcal{E}_1, j_*\mathcal{E}_2) \simeq R\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \simeq \begin{cases} \mathbb{C} & \text{if } (\mathcal{O}_1, \mathcal{E}_1) \simeq (\mathcal{O}_2, \mathcal{E}_2) \\ 0 & \text{otherwise.} \end{cases}$$

Putting these facts together we obtain the decomposition

$$\mathbf{D}(\mathfrak{l})_{cusp} = \bigoplus_{(c\mathcal{O}, \mathcal{E})} \mathbf{D}(\mathfrak{l})_{(\mathcal{O}, \mathcal{E})},$$

where $\mathbf{D}(\mathfrak{l})_{(\mathcal{O}, c\mathcal{E})}$ is the full subcategory whose objects are the form $\mathfrak{M} \boxtimes IC(\mathcal{O}, \mathcal{E})$. Note that

$$\mathbf{D}(\mathfrak{l})_{(\mathcal{O}, \mathcal{E})} \simeq \mathbf{D}(\mathfrak{z}(\mathfrak{l})) \simeq \mathfrak{D}(\mathfrak{z}(\mathfrak{l})) \otimes \Lambda_{\mathfrak{z}(\mathfrak{l})} - \mathbf{dgMod}.$$

We define $\mathfrak{D}(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$ to be the subcategory consisting of objects \mathfrak{M} such that $\mathbf{Res}_{P, L}^G(\mathfrak{M})$ is an object of $\mathbf{D}(\mathfrak{l})_{(\mathcal{O}, \mathcal{E})}$. It follows that $\mathbf{D}(\mathfrak{g})_{(L)}$ decomposes as an orthogonal sum of subcategories $\mathbf{D}(\mathfrak{g})_{(L, \mathcal{O}, \mathcal{E})}$. This completes the proof of Theorem 0.3.

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